November 2012 DFUB 12

AKSZ MODELS OF SEMISTRICT HIGHER GAUGE THEORY

by

Roberto Zucchini

Dipartimento di Fisica, Università degli Studi di Bologna

V. Irnerio 46, I-40126 Bologna, Italy

I.N.F.N., sezione di Bologna, Italy

E–mail: zucchinir@bo.infn.it

Abstract

In the first part of this paper, we work out a perturbative Lagrangian formulation of semistrict higher gauge theory, that avoids the subtleties of the relationship between Lie 2–groups and algebras by relying exclusively on the structure semistrict Lie 2–algebra $\mathfrak{v}$ and its automorphism 2–group $\mathrm{Aut}(\mathfrak{v})$. Gauge transformations are defined and shown to form a strict 2–group depending on $\mathfrak{v}$. Fields are $\mathfrak{v}$–valued and their global behaviour is controlled by appropriate gauge transformation gluing data organized as a strict 2–groupoid. In the second part, using the BV quantization method in the AKSZ geometrical version, we write down a 3–dimensional semistrict higher BF gauge theory generalizing ordinary BF theory, carry out its gauge fixing and obtain as end result a semistrict higher topological gauge field theory of the Witten type. We also introduce a related 4–dimensional semistrict higher Chern–Simons gauge theory.

Keywords: quantum field theory in curved space–time; geometry, differential geometry and topology. PACS: 04.62.+v 02.40.-k

Contents

1 Introduction

Higher gauge theory is a generalization of ordinary gauge theory in which the gauge potentials have form degree $p \geq 1$ and, correspondingly, their gauge curvatures have degree $p+1 \geq 2$. Parallel transport, as a consequence, is defined along p–dimensional submanifolds. Higher gauge theory can be both Abelian and non Abelian and may require several gauge potentials of different form degree for its consistency.

The origin of Abelian higher gauge theory can be traced back to the early days of supergravity. Expectedly, Abelian higher gauge theory enters also in string and M–theory, which have supergravity as their low energy energy limit [1, 2]. The Kalb–Ramond B–field of the 10–dimensional type II theories is a 2–form field governed by a higher form of electrodynamics, 2–form electrodynamics, in which fundamental strings act as sources. Likewise, the Ramond-Ramond fields are p–form fields described by p–form electrodynamics whose sources are D–branes. Analogously, the 11–dimensional theory C–field is a 3–form field described by 3–form electrodynamics and sourced by $M2$ branes.

The physics of stacks of coinciding branes is encoded in non Abelian higher gauge theory [3]. While stacks of D–branes are governed by ordinary Yang–Mills theory, those of $M5$–branes are believed to be described by a higher non Abelian gauge theory whose details are still not completely understood.

Non Abelian higher gauge theory, especially in its integral version, has also found application in the theories of quantum gravity alternative to string theory such as loop quantum gravity and, in particular, spin foam models [4, 5].

In general, higher gauge theory is expected to be play a basic role whenever charged higher–dimensional extended objects are involved. For this reason, higher gauge theory has been the object of independent analysis since quite early [6–12]. However, its intensive study began only in the last decade or so [13–18]. See

ref. [19] for an up–to–date review of this subject and extensive referencing.

From a mathematical point of view, higher gauge theory is intimately related to higher algebraic structures, such as 2–categories, 2–groups [20, 21] and strong homotopy Lie or L_∞ algebras [22, 23] and higher geometrical structures such as gerbes [24, 25].

Since its inception, higher gauge theory has always had a topological flavour. In fact, higher gauge theory has intersected topological field theory at relevant points. In particular, the so called BF theory [26, 27] and its variants have played an important part [28–31].

Higher gauge theory has been developed by promoting the basic topological and geometrical structures of ordinary gauge theory to a higher level in category theory by a procedure called *internalization* [32, 33]. In the same way as ordinary gauge theory relies on Lie groups, Lie algebras and principal bundles over manifolds with connections, higher gauge theory does on their next level categorified counterparts, viz Lie 2–groups, Lie 2–algebras and principal 2–bundles on 2–manifolds with 2–connections. This outline, nicely intuitive as it is, hides the complexity of the matter however. Lie 2–groups and 2–algebras come in two broad varieties. They are either *strict*, when the basic Lie group and algebra theoretic relations hold strictly as identities, or *non strict*, when those relations are allowed to hold only up to isomorphism. In the non strict case, the so–called *weak* or *coherent* 2—*groups* and *semistrict Lie* 2*–algebras* have been studied, e. g. in [20, 21]. We thus distinguish strict Lie 2–group based higher gauge theory, henceforth referred to as *strict higher gauge theory*, from coherent Lie 2–group based higher gauge theory, which we christen *semistrict higher gauge theory*. While there exists a large body of literature about the strict theory, a comparatively smaller amount of work has been devoted to the study and exemplification of the semistrict theory. The present paper is a further modest contribution to the latter. We are going to propose a partially original formulation of semistrict

higher gauge theory, candidly assessing its merits and shortcomings and illustrating it with a number of sample calculations. A summary description of it is given next.

Simply speaking, a *semistrict Lie 2–algebra* is a 2–vector space $\mathfrak{v}$ equipped with a bilinear functor $[\cdot,\cdot] : \mathfrak{v} \times \mathfrak{v} \mapsto \mathfrak{v}$, the Lie bracket, that is antisymmetric and satisfies the Jacobi identity up to a possibly trivial completely antisymmetric trilinear natural isomorphism, the *Jacobiator*, which in turn is required to satisfy a certain coherence relation, the *Jacobiator identity*.

The finite counterpart of a Lie 2–algebra is a 2–group. A *2–group* is a category equipped with a multiplication, a unit and an inversion functor analogous to group operations but satisfying the associativity, unit and inverse law only up to possibly trivial natural isomorphisms satisfying coherence relations. A *Lie 2–group* is a 2 group, whose underlying category is a smooth one.

An *L_∞ algebra* is a chain complex $\mathfrak{v}$ of vector spaces equipped with a bilinear antisymmetric operation $[\cdot,\cdot] : \mathfrak{v} \times \mathfrak{v} \mapsto \mathfrak{v}$, which satisfies the Jacobi identity up to an infinite tower of chain homotopies. When the complex is non trivial only in degree $0, \dots, n-1$, we have an *n–term L_∞ algebra*. 2–term L_∞ algebras and semistrict Lie 2–algebras constitute 2–categories which can be shown to be equivalent in the appropriate sense. Therefore, we can formulate the theory of semistrict Lie 2–algebras in the language of that of 2–term L_∞ algebras. This is indeed the way we shall proceed and, for this reason, in the following we shall often refer to semistrict higher gauge theory as *2–term L_∞ algebra gauge theory*. The basic notions and properties of 2–term L_∞ algebras and their manifold relations to Lie 2–groups are reviewed in sect. 2.

A conventional formulation of semistrict gauge theory modelled on that of ordinary gauge theory along the lines of refs. [32, 33] (see also ref. [34]) would presumably require a principal 2–bundle $P(M, V)$ on a 2–manifold M with a structure coherent 2–group V. The structure Lie 2–algebra $\mathfrak{v}$ would be only a

derived secondary object. (For the sake of simplicity, but oversimplifying a bit, we leave aside the subtle issues involved in the relationship between V and $\mathfrak{v}$ in the non strict case.) This type of approach is the most powerful in theory, but its concrete implementation appears to be presently beyond our reach in practice and, for this reason, we follow another route.

Consider an ordinary gauge theory with structure group G. The topological background of the theory is then a principal G–bundle P represented by an equivalence class of G–valued 1–cocycles $\gamma = \{\gamma_{ij}\}$ with respect to an open covering $U = \{U_i\}$ of the base manifold M. Since in gauge theory all fields are in the adjoint of G, the effective structure group is the adjoint group $\mathrm{Ad}\, G = G/Z(G)$ rather than G. Can we, then, replace G by $\mathrm{Ad}\, G$ in our gauge theory? The answer to this question is positive if, from the knowledge of the data $g_{ij} = \mathrm{Ad}\,\gamma_{ij}$, it is possible to reconstruct the data $\sigma_{ij} = {\gamma_{ij}}^{-1} d\gamma_{ij}$ which control the global definition of the gauge fields. This can be done only G is semisimple, e. g. $G = \mathrm{SU}(n)$. However, we can still work with $\mathrm{Ad}\, G$ rather than G, if we give up the condition that the σ_{ij} be determined by the g_{ij} and regard the g_{ij} and σ_{ij} as a whole set of data satisfying the relations

$$d\sigma_{ij} + \frac{1}{2}[\sigma_{ij}, \sigma_{ij}] = 0, \tag{1.0.1a}$$

$${g_{ij}}^{-1} dg_{ij}(x) - [\sigma_{ij}, x] = 0, \qquad x \in \mathfrak{g}, \tag{1.0.1b}$$

together the cocycle conditions

$$g_{ij} g_{jk} = g_{ik}, \tag{1.0.2a}$$

$$\sigma_{ik} - \sigma_{jk} - {g_{jk}}^{-1}(\sigma_{ij}) = 0. \tag{1.0.2b}$$

As $\mathrm{Ad}\, G \simeq \mathrm{Inn}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$, however, proceeding in this way we are generalizing gauge theory, since now g_{ij} is allowed to take values in the full automorphism group $\mathrm{Aut}(\mathfrak{g})$ rather than the inner one $\mathrm{Inn}(\mathfrak{g})$.

This leads to a formulation of gauge theory that can be summarized as fol-

lows. The basic datum is a finite dimensional structure Lie algebra $\mathfrak{g}$. At finite level, instead of a Lie group G integrating $\mathfrak{g}$, we rely on the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. A gauge transformation on a neighborhood O consists of a map $g \in \mathrm{Map}(O, \mathrm{Aut}(\mathfrak{g}))$ together with a flat connection σ on O satisfying certain relations. Gauge transformations form an infinite dimensional group $\mathrm{Gau}(O, \mathfrak{g})$. A left action of $\mathrm{Gau}(O, \mathfrak{g})$ on fields on O is defined. Given an open covering $U = \{U_i\}$, the global definedness of the fields is controlled by a $\mathrm{Gau}(\cdot, \mathfrak{g})$–valued cocycle, which comprises an $\mathrm{Aut}(\mathfrak{g})$–valued cocycle $\{g_{ij}\}$ and a set of flat connection data $\{\sigma_{ij}\}$ satisfying (1.0.1), (1.0.2). These latter constitute the 0–cells of a groupoid $\check{\mathcal{P}}(U, \mathfrak{g})$ which describes the underlying topology.

The theory, so, can be formulated to a large extent relying on the Lie algebra $\mathfrak{g}$ only. It is clear that the gauge theoretic framework outlined above can only work in *perturbative Lagrangian field theory*. As it is, it is unsuitable for the analysis of parallel transport, a central issue in gauge theory. Further, as it is well–known, important non perturbative effects are attached to the center $Z(G)$ of G, information about which is lost. The reason why the approach is nevertheless useful is that it can be directly generalized to semistrict higher gauge theory.

Our formulation of semistrict higher gauge theory follows basically the same lines. The basic datum is a finite dimensional structure Lie 2–algebra $\mathfrak{v}$, conveniently viewed as a 2–term L_∞ algebra. At finite level, instead of a Lie 2–group V integrating $\mathfrak{v}$, which may be infinite dimensional or may be something more general than a mere coherent 2–group, we rely on the automorphism 2–group $\mathrm{Aut}(\mathfrak{v})$ of $\mathfrak{v}$, which is always finite dimensional and strict. A gauge transformation on a neighborhood O consists of a map $g \in \mathrm{Map}(O, \mathrm{Aut}(\mathfrak{v}))$ together with a flat connection doublet (σ, Σ) on O and other form data satisfying a set of relations. Gauge transformations form an infinite dimensional group $\mathrm{Gau}_1(O, \mathfrak{v})$, which is the 1–cell group of a strict 2–group $\mathrm{Gau}(O, \mathfrak{v})$. A left action of $\mathrm{Gau}_1(O, \mathfrak{v})$ on fields on O is defined. Given an open covering $U = \{U_i\}$, the global defined-

ness of the fields is controlled by a $\mathrm{Gau}(\cdot, \mathfrak{g})$–valued cocycle, which comprises an $\mathrm{Aut}(\mathfrak{v})$–valued cocycle $\{g_{ij}, W_{ijk}\}$, a set of flat connection doublet data $\{\sigma_{ij}, \Sigma_{ij}\}$ and other form data satisfying relations generalizing (1.0.1), (1.0.2). These latter constitute the 0–cells of a strict 2–groupoid $\check{\mathcal{P}}_2(U, \mathfrak{v})$ which describes the underlying topology. Given its novelty and its relevance in the subsequent constructions, this matter is expounded in great detail in sect. 3.

The strict case where $\mathfrak{v}$ is the differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$ associated with a Lie crossed module (G, H), which is widely discussed in the literature and very well understood, can be described in our framework. Indeed, one can show that $i)$ gauge transformations on O, as customarily defined in this case (see e. g. [32,33]), can be organized in an infinite dimensional strict 2–group $\mathrm{Gau}(O, G, H)$ and $ii)$ that there is a natural 2–group morphism to $\mathrm{Gau}(O, G, H) \rightarrow \mathrm{Gau}(O, \mathfrak{v})$, which translates the familiar strict higher gauge theoretic framework into ours. A recent very general formulation of higher gauge theory, proposed in refs. [35,36], is also related to ours, though non manifestly so. See again sect. 3.

Our approach has its advantages and disadvantages. On the differential side, it is very efficient and provides a powerful algorithm for the construction of local semistrict higher gauge models in perturbative Lagrangian field theory. On the integral side, as its counterpart of ordinary gauge theory, it is apparently not suitable for the study and efficient computation of higher parallel transport, even in the strict theory. With its admitted limitations, this is anyway the line of thought we follow.

With the suitable differential geometric tools available to us, the construction of semistrict higher gauge field theories becomes possible. Indeed, there is an elegant methodology for working out consistent quantum field theories relying only on a given set of differential geometric data based on the Batalin–Vilkovisky (BV) quantization algorithm [37,38] and known as Alexandrov–Kontsevich–Schwartz–Zaboronsky (AKSZ) construction [39,40]. Following this path and borrowing

ideas from previous work [41, 42], we are able to write down a 3–dimensional semistrict higher BF gauge theory, carry out its gauge fixing and obtain eventually a semistrict higher topological gauge theory of the Witten type [43]. The BF gauge field theory we get differs at significant points from the ones which have appeared in the literature mentioned earlier and is interesting on its own. We also outline briefly a related 4–dimensional semistrict higher Chern–Simons gauge theory [44], here understood as a gauge theory whose field equations are flatness conditions. These models are illustrated in sect. 4. We have found that they belong to the class of models covered by the general analysis of though not explicitly studied in refs. [45, 46].

Many problems remain open and many issues require further investigation. They are discussed briefly in sect. 5.

Finally, in the appendices, we provide explicit formulae for the action and BV symmetry variations in components for the BV field theories studied in the main body of the paper.

2 Lie 2–algebras and 2–groups

The symmetries of higher gauge theory are believed to be encoded in Lie 2–algebras and 2–groups [19]. So, we may begin our discussion from these. Below, we review the basic notions of the theory of Lie 2–algebras and 2–groups, a subject still largely unknown among non experts. Our presentation is admittedly incomplete, leaving as it does the important categorical aspects in the background, and occasionally lacking in mathematical rigour. An exhaustive treatment would go beyond the scope of the present paper. The one given below furnishes the reader with all the basic definitions and results required for the understanding of the second half of the paper. It also sets our notation and terminology once and for all. Though with limitations, it is tailor made for our purposes.

2.1 Lie 2–algebras

Lie 2–algebras are the next higher analog of ordinary Lie algebras. Let us recall briefly the definition of this notion. A *2–vector space* is a category internal to the category Vect of vector spaces, that is a category whose objects and morphisms are vector spaces and whose source, target, identity and composition maps are all linear [21]. A *semistrict Lie 2–algebra*, or more concisely a Lie 2–algebra, is a 2–vector space equipped with a bilinear and antisymmetric bracket functor, which satisfies the Jacobi identity up to a natural isomorphism, called the Jacobiator. This latter in turn satisfies a coherence law, the Jacobiator identity.

In [21], it is shown that there is a one–to–one correspondence between equivalence classes of Lie 2–algebras and isomorphism classes of the following data:

1. a Lie algebra $\mathfrak{g}$;

2. an Abelian Lie algebra $\mathfrak{h}$;

3. a homomorphism $\rho : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{h})$;

4. an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$ of the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$.

The correspondence hinges on the close relationship between Lie 2–algebras and 2–term L_∞ algebras, as we explain next.

L_∞ algebras were originally introduced by Schlessinger and Stasheff in [47]. Since then, they have found several applications in field and string theory. (See [22] for a readable self contained account.) A L_∞ algebra is a higher generalization of a Lie algebra, in which the Jacobi identity holds only up to higher coherent homotopy. A *semistrict 2–term L_∞ algebra*, or more briefly a 2–term L_∞ algebra, is a special, particularly simple kind of L_∞ algebra. It generalizes a Lie algebra and a differential Lie crossed module by allowing the Lie bracket to have a non trivial Jacobiator. In [21], it is proven that Lie 2–algebras form a 2–category which is 2–equivalent to the 2–category of 2–term L_∞ algebras. In practice this means that we can think of every Lie 2–algebra equivalently as a 2–term L_∞ algebra.

The proof of the classification theorem quoted above in outline goes as follows. First, one proves that a given Lie 2–algebra $\mathfrak{v}$ is equivalent to a *skeletal* Lie 2–algebra $\mathfrak{v}_s$, that is one in which all isomorphic objects are equal. Next, one demonstrates that under the equivalence of the categories of Lie 2–algebras and 2–term L_∞ algebras, the isomorphism classes of skeletal Lie 2–algebras are in one–to–one correspondence with those of 2–term L_∞ algebras with vanishing differential. Finally, one shows that these latter classes are in one–to–one correspondence with the isomorphism classes of the above data.

Though from a categorical point of view is more natural to work with Lie 2–algebras, in field theoretic applications it is definitely more convenient to deal with 2–term L_∞ algebras, because these lend themselves to rather explicit calculations. We thus turn to them.

2.2 2–term L_∞ algebras

A 2–*term* L_∞ *algebra* consists of the following set of data:

1. a pair of vector spaces on the same field $\mathfrak{v}_0, \mathfrak{v}_1$;
2. a linear map $\partial : \mathfrak{v}_1 \to \mathfrak{v}_0$;
3. a linear map $[\cdot,\cdot] : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_0$;
4. a linear map $[\cdot,\cdot] : \mathfrak{v}_0 \otimes \mathfrak{v}_1 \to \mathfrak{v}_1$;
5. a linear map $[\cdot,\cdot,\cdot] : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_1$ [1].

These are required to satisfy the following axioms:

$$[x, \partial X] - \partial[x, X] = 0, \tag{2.2.1a}$$

$$[\partial X, Y] + [\partial Y, X] = 0, \tag{2.2.1b}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] - \partial[x, y, z] = 0, \tag{2.2.1c}$$

$$[x, [y, X]] - [y, [x, X]] - [[x, y], X] - [x, y, \partial X] = 0, \tag{2.2.1d}$$

$$\begin{aligned}&[x, [y, z, w]] - [y, [z, w, x]] + [z, [w, x, y]] - [w, [x, y, z]] \\ &- [x, y, [z, w]] - [x, z, [w, y]] - [x, w, [y, z]] \\ &+ [y, z, [w, x]] + [w, y, [z, x]] + [z, w, [y, x]] = 0,\end{aligned} \tag{2.2.1e}$$

where $x, y, z, \ldots \in \mathfrak{v}_0$, $X, Y, Z, \ldots \in \mathfrak{v}_1$ here and below. In the following, we shall denote a 2–term L_∞ algebra such as the above by $\mathfrak{v}$ or, more explicitly, by $(\mathfrak{v}_0, \mathfrak{v}_1, \partial, [\cdot,\cdot], [\cdot,\cdot,\cdot])$ to emphasize its underlying structure. When dealing with several such algebras, we shall use apexes to distinguish them.

A 2–term L_∞ algebra $\mathfrak{v}$ is frequently thought of as a 2–term chain complex $0 \longrightarrow \mathfrak{v}_1 \xrightarrow{\partial} \mathfrak{v}_0 \longrightarrow 0$ (whence its name) equipped with a degree 0 graded

[1] We denote by $[\cdot,\cdot]$ both 2–argument brackets. It will be clear from context which is which.

antisymmetric bilinear bracket $[\cdot,\cdot]$ and a degree 1 graded antisymmetric trilinear bracket $[\cdot,\cdot,\cdot]$ enjoying the following properties. First, the boundary ∂ satisfies the graded Leibniz identity with respect to $[\cdot,\cdot]$ (cf. eqs. (2.2.1a), (2.2.1b)). Second, $[\cdot,\cdot]$ does not satisfy the graded Jacobi identity and, so, is not a Lie bracket in general. The Jacobiator of $[\cdot,\cdot]$ equals the Leibnizator of ∂ with respect to $[\cdot,\cdot,\cdot]$ (cf. eqs. (2.2.1c), (2.2.1d)). Third, the brackets $[\cdot,\cdot]$, $[\cdot,\cdot,\cdot]$ must satisfy a certain consistency condition (cf. eq. (2.2.1e)). Thus, $\mathfrak{v}$ is a graded differential Lie algebra up to coherent homotopy (whence the name 2–term strong homotopy Lie algebra frequently used.)

2.3 2–term L_∞ algebra morphisms

The notion of 2–term L_∞ algebra morphism is expected to play an important role in the theory of 2–term L_∞ algebras on general grounds. It also underlies the basic fact that 2–term L_∞ algebras form a 2–category. Our treatment follows closely that given in [21].

Let $\mathfrak{v}$, $\mathfrak{v}'$ be 2–term L_∞ algebras. A *2–term L_∞ algebra 1–morphism* from $\mathfrak{v}$ to $\mathfrak{v}'$ consists of the following data:

1. a vector space morphism $\phi_0 : \mathfrak{v}_0 \to \mathfrak{v}'_0$;
2. a vector space morphism $\phi_1 : \mathfrak{v}_1 \to \mathfrak{v}'_1$;
3. a vector space morphism $\phi_2 : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}'_1$.

These are required to satisfy the following relations:

$$\phi_0(\partial X) - \partial'\phi_1(X) = 0, \tag{2.3.1a}$$

$$\phi_0([x,y]) - [\phi_0(x),\phi_0(y)]' - \partial'\phi_2(x,y) = 0, \tag{2.3.1b}$$

$$\phi_1([x,X]) - [\phi_0(x),\phi_1(X)]' - \phi_2(x,\partial X) = 0, \tag{2.3.1c}$$

$$[\phi_0(x), \phi_2(y,z)]' + [\phi_0(y), \phi_2(z,x)]' + [\phi_0(z), \phi_2(x,y)]' + \phi_2(x,[y,z]) \qquad (2.3.1d)$$
$$+ \phi_2(y,[z,x]) + \phi_2(z,[x,y]) - \phi_1([x,y,z]) + [\phi_0(x), \phi_0(y), \phi_0(z)]' = 0.$$

In the following, we shall denote a 1–morphism such as the above one by ϕ or, more explicitly, by (ϕ_0, ϕ_1, ϕ_2) to emphasize its constituent components. We shall also write $\phi : \mathfrak{v} \to \mathfrak{v}'$ to indicate the source and target algebras of ϕ.

To make the notion of 1–morphism more transparent, it is necessary to think of $\mathfrak{v}$ as a 2–term chain complex $0 \longrightarrow \mathfrak{v}_1 \xrightarrow{\partial} \mathfrak{v}_0 \longrightarrow 0$ equipped with a degree 0 graded antisymmetric bilinear bracket $[\cdot,\cdot]$ and a degree 1 graded antisymmetric trilinear bracket $[\cdot,\cdot,\cdot]$ with certain properties and similarly for $\mathfrak{v}'$ (cf. sect. 2.2). If $\phi : \mathfrak{v} \to \mathfrak{v}'$ is a 1–morphism, then the maps ϕ_0, ϕ_1 are the components of a chain map of $\tilde{\phi} : \mathfrak{v} \to \mathfrak{v}'$ (cf. eq. (2.3.1a)). Further, $[\tilde{\phi}(\cdot), \tilde{\phi}(\cdot)], \tilde{\phi}([\cdot,\cdot]) : \mathfrak{v} \otimes \mathfrak{v} \to \mathfrak{v}'$ are chain maps, $\mathfrak{v} \otimes \mathfrak{v}$ being the tensor square of the chain complex $\mathfrak{v}$, and ϕ_2 is a chain homotopy of such chain maps (cf. eqs. (2.3.1b), (2.3.1c)). Finally, ϕ_0, ϕ_1, ϕ_2 must satisfy a coherence relation ensuring their compatibility with the basic relations (2.2.1) satisfied by the brackets of $\mathfrak{v}$ (cf. eq. (2.3.1d)).

For any two 2–term L_∞ algebra 1–morphisms $\phi, \psi : \mathfrak{v} \to \mathfrak{v}'$, a *2–term L_∞ algebra 2–morphism* from ϕ to ψ consists of a single datum:

1. a linear map $\Phi : \mathfrak{v}_0 \to \mathfrak{v}'_1$.

This must satisfy the following relations

$$\phi_0(x) - \psi_0(x) = \partial' \Phi(x), \qquad (2.3.2a)$$

$$\phi_1(X) - \psi_1(X) = \Phi(\partial X), \qquad (2.3.2b)$$

$$\phi_2(x,y) - \psi_2(x,y) + [\phi_0(x), \Phi(y)]' - [\psi_0(y), \Phi(x)]' - \Phi([x,y]) = 0. \qquad (2.3.2c)$$

We shall write a 2–morphism such as this as Φ or as $\Phi : \phi \Rightarrow \psi$ to emphasize its source and target.

To clarify the notion of 2–morphism, one must regard again $\mathfrak{v}$ as a 2–term chain complex $0 \longrightarrow \mathfrak{v}_1 \xrightarrow{\partial} \mathfrak{v}_0 \longrightarrow 0$ equipped with a graded antisymmetric multilinear brackets $[\cdot,\cdot]$, $[\cdot,\cdot,\cdot]$ and similarly $\mathfrak{v}'$ and view $\phi,\psi:\mathfrak{v}\to\mathfrak{v}'$ as chain maps $\tilde{\phi},\tilde{\psi}:\mathfrak{v}\to\mathfrak{v}'$ as explained earlier. Then, a 2–morphim $\varPhi:\phi\Rightarrow\psi$ is a chain homotopy of the chain maps $\tilde{\phi},\tilde{\psi}$. (cf. eqs. (2.3.2a), (2.3.2b)). $\varPhi$ must satisfy further a coherence relation ensuring its compatibility with the basic relations (2.3.1) satisfied by ϕ,ψ (cf. eq. (2.3.2c)).

Next, we shall define a composition law and a unit for 1–morphisms and horizontal and vertical composition laws and units for 2–morphisms.

The composition of two 2–term L_∞ algebra 1–morphisms $\phi:\mathfrak{v}\to\mathfrak{v}'$, $\psi:\mathfrak{v}'\to\mathfrak{v}''$ is the 1–morphism $\psi\circ\phi:\mathfrak{v}\to\mathfrak{v}''$ defined componentwise by

$$(\psi\circ\phi)_0(x)=\psi_0\phi_0(x), \tag{2.3.3a}$$

$$(\psi\circ\phi)_1(X)=\psi_1\phi_1(X), \tag{2.3.3b}$$

$$(\psi\circ\phi)_2(x,y)=\psi_1\phi_2(x,y)+\psi_2(\phi_0(x),\phi_0(y)). \tag{2.3.3c}$$

For any 2–term L_∞ algebra $\mathfrak{v}$, the identity of $\mathfrak{v}$ is the 2–term L_∞ algebra 1–morphism $\mathrm{id}_\mathfrak{v}:\mathfrak{v}\to\mathfrak{v}$ defined componentwise by

$$\mathrm{id}_{\mathfrak{v}0}(x)=x, \tag{2.3.4a}$$

$$\mathrm{id}_{\mathfrak{v}1}(X)=X, \tag{2.3.4b}$$

$$\mathrm{id}_{\mathfrak{v}2}(x,y)=0. \tag{2.3.4c}$$

The horizontal composition of two 2–term L_∞ algebra 2–morphisms $\varPhi:\lambda\Rightarrow\mu$, $\varPsi:\phi\Rightarrow\psi$, with $\lambda,\mu:\mathfrak{v}\to\mathfrak{v}'$, $\phi,\psi:\mathfrak{v}'\to\mathfrak{v}''$ 2–term L_∞ algebra 1–morphisms, is the 2–morphism $\varPsi\circ\varPhi:\phi\circ\lambda\Rightarrow\psi\circ\mu$ defined by

$$\varPsi\circ\varPhi(x)=\varPsi\lambda_0(x)+\psi_1\varPhi(x)=\varPsi\mu_0(x)+\phi_1\varPhi(x). \tag{2.3.5}$$

The vertical composition of two 2–term L_∞ algebra 2–morphisms $\varPi:\lambda\Rightarrow\mu$,

$\Lambda : \mu \Rightarrow \nu$, with $\lambda, \mu, \nu : \mathfrak{v} \to \mathfrak{v}'$ 2–term L_∞ algebra 1–morphisms, is the 2–morphism $\Lambda \boldsymbol{\cdot} \Pi : \lambda \Rightarrow \nu$ defined by

$$\Lambda \boldsymbol{\cdot} \Pi(x) = \Pi(x) + \Lambda(x). \tag{2.3.6}$$

Finally, for any 2–term L_∞ algebra 1–morphism $\phi : \mathfrak{v} \to \mathfrak{v}'$, the identity of ϕ is the 2–term L_∞ algebra 2–morphism $\mathrm{Id}_\phi : \phi \Rightarrow \phi$ given by

$$\mathrm{Id}_\phi(x) = 0. \tag{2.3.7}$$

The composition of 1–morphisms, the unit of an L_∞ algebra, the horizontal and vertical composition of 2–morphisms and the unit of a 1–morphism satisfy the following basic relations

$$(\nu \circ \mu) \circ \lambda = \nu \circ (\mu \circ \lambda), \tag{2.3.8a}$$

$$\lambda \circ \mathrm{id}_{\mathfrak{v}} = \mathrm{id}_{\mathfrak{v}'} \circ \lambda = \lambda, \tag{2.3.8b}$$

$$(\Lambda \circ \Psi) \circ \Phi = \Lambda \circ (\Psi \circ \Phi), \tag{2.3.8c}$$

$$\Phi \circ \mathrm{Id}_{\mathrm{id}_{\mathfrak{v}}} = \mathrm{Id}_{\mathrm{id}_{\mathfrak{v}'}} \circ \Phi = \Phi, \tag{2.3.8d}$$

$$(\Lambda \boldsymbol{\cdot} \Psi) \boldsymbol{\cdot} \Phi = \Lambda \boldsymbol{\cdot} (\Psi \boldsymbol{\cdot} \Phi), \tag{2.3.8e}$$

$$\Phi \boldsymbol{\cdot} \mathrm{Id}_\lambda = \mathrm{Id}_\mu \boldsymbol{\cdot} \Phi = \Phi, \tag{2.3.8f}$$

$$(\Theta \boldsymbol{\cdot} \Lambda) \circ (\Psi \boldsymbol{\cdot} \Phi) = (\Theta \circ \Psi) \boldsymbol{\cdot} (\Lambda \circ \Phi), \tag{2.3.8g}$$

holding whenever the various instances of morphism composition are defined. (2.3.8) are precisely the relations which render the class of 2–term L_∞ algebras a (strict) 2–category. This fact has a great mathematical salience, though it will matter only marginally in the field theoretic applications treated later.

With the appropriate 2–term L_∞-algebra morphism structure at one's disposal, it is possible to define the notion of equivalence of 2–term L_∞-algebras. Two such algebras $\mathfrak{v}$, $\mathfrak{v}'$ are said *equivalent* if there are 1–morphisms $\phi : \mathfrak{v} \to \mathfrak{v}'$, $\psi : \mathfrak{v}' \to \mathfrak{v}$ and vertically invertible 2–morphisms $\Phi : \psi \circ \phi \Rightarrow \mathrm{id}_{\mathfrak{v}}$ and $\Psi : \phi \circ \psi \Rightarrow$

$\mathrm{id}_{\mathfrak{v}'}$ [2]. Isomorphism implies equivalence but not viceversa.

2.4 Strict Lie 2–algebras and differential Lie crossed modules

Strict Lie 2–algebras form a distinguished subclass of the class of Lie 2–algebras, which is well understood and appears in many important applications. Further, they are intimately related to differential Lie crossed modules.

A 2–term L_∞ algebra $(\mathfrak{v}_0, \mathfrak{v}_1, \partial, [\cdot,\cdot], [\cdot,\cdot,\cdot])$ is *strict* if $[\cdot,\cdot,\cdot] = 0$ identically.

Inspecting (2.2.1), we realize that then $\mathfrak{v}_0$ is an ordinary Lie algebra, $\mathfrak{v}_1$ is a $\mathfrak{v}_0$ Lie module and ∂ is a Casimir for the latter.

A *differential Lie crossed module* [48] consists in the following elements.

1. A pair of Lie algebras $\mathfrak{g}$, $\mathfrak{h}$.

2. A Lie algebra morphism $\tau : \mathfrak{h} \to \mathfrak{g}$.

3. A Lie algebra morphism $\mu : \mathfrak{g} \to \mathfrak{der}(\mathfrak{h})$, where $\mathfrak{der}(\mathfrak{h})$ is the Lie algebra of derivations of $\mathfrak{h}$.

Further, the following conditions are verified,

$$\tau(\mu(x)(X)) = [x, \tau(X)]_{\mathfrak{g}}, \tag{2.4.1a}$$

$$\mu(\tau(X))(Y) = [X, Y]_{\mathfrak{h}}, \tag{2.4.1b}$$

where $x, y, \ldots \in \mathfrak{g}$, $X, Y, \cdots \in \mathfrak{h}$. We shall denote a differential Lie crossed module such as this by $(\mathfrak{g}, \mathfrak{h})$ or $(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$ to explicitly indicate its underlying structure.

There exists a one–to–one correspondence between strict 2–term L_∞ algebras and differential Lie crossed modules. With any differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$, there is associated a strict 2–term L_∞ algebra $\mathfrak{v}$ as follows.

[2] A 2–morphism $\Lambda : \lambda \Rightarrow \mu$ is said vertically invertible if there is a 2–morphism $\Pi : \mu \Rightarrow \lambda$ such that $\Pi \cdot \Lambda = \mathrm{Id}_\lambda$, $\Lambda \cdot \Pi = \mathrm{Id}_\mu$. Φ, Ψ here can be shown to be automatically vertically invertible.

1. $\mathfrak{v}_0 = \mathfrak{g}$;

2. $\mathfrak{v}_1 = \mathfrak{h}$;

3. $\partial X = \tau(X)$;

4. $[x, y] = [x, y]_{\mathfrak{g}}$;

5. $[x, X] = \mu(x)(X)$;

6. $[x, y, z] = 0$.

Conversely, with any strict 2–term L_∞ algebra $\mathfrak{v}$, there is associated a differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$ as follows.

1. $\mathfrak{g} = \mathfrak{v}_0$;

2. $\mathfrak{h} = \mathfrak{v}_1$;

3. $[x, y]_{\mathfrak{g}} = [x, y]$;

4. $[X, Y]_{\mathfrak{h}} = [\partial X, Y]$;

5. $\tau(X) = \partial X$;

6. $\mu(x)(X) = [x, X]$.

Let $\mathfrak{v}$, $\mathfrak{v}'$ be strict 2–term L_∞ algebras. A *strict 2–term L_∞ algebra 1–morphism* from $\mathfrak{v}$ to $\mathfrak{v}'$ is a 2–term L_∞ algebra 1–morphism $\phi : \mathfrak{v} \to \mathfrak{v}'$ such that $\phi_2 = 0$. For two strict 2–term L_∞ algebra 1–morphisms $\phi, \psi : \mathfrak{v} \to \mathfrak{v}'$, a strict 2–term L_∞ algebra 2–morphism from ϕ to ψ is an ordinary 2–term L_∞ algebra 2–morphism $\Phi : \phi \Rightarrow \psi$. With the strict morphism structure defined above, strict 2–term L_∞ algebras form a sub–2–category of the 2–category of 2–term L_∞ algebras.

Let $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}', \mathfrak{h}')$ be differential Lie crossed modules. A *differential Lie crossed module morphism* from $(\mathfrak{g}, \mathfrak{h})$ to $(\mathfrak{g}', \mathfrak{h}')$ is a pair of

1. a Lie algebra morphism $\beta : \mathfrak{g} \to \mathfrak{g}'$,

2. a Lie algebra morphism $\gamma : \mathfrak{h} \to \mathfrak{h}'$

preserving the crossed module relations,

$$\beta(\tau(X)) = \tau'(\gamma(X)), \tag{2.4.2a}$$

$$\gamma(\mu(x)(X)) = \mu'(\beta(x))(\gamma(X)). \tag{2.4.2b}$$

We shall denote a crossed module morphism like the above as (β, γ) or $(\beta, \gamma) : (\mathfrak{g}, \mathfrak{h}) \to (\mathfrak{g}', \mathfrak{h}')$.

With the morphism structure just defined, differential Lie crossed modules form a category.

There is a obvious one–to–one correspondence between strict 2–term L_∞ algebra 1–morphisms $\phi : \mathfrak{v} \to \mathfrak{v}'$ and crossed module morphism $(\beta, \gamma) : (\mathfrak{g}, \mathfrak{h}) \to (\mathfrak{g}', \mathfrak{h}')$, obtained by viewing the strict 2–term L_∞ algebras $\mathfrak{v}$, $\mathfrak{v}'$ as the differential Lie crossed modules $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}', \mathfrak{h}')$, as described above. Explicitly,

1. $\phi_0(x) = \beta(x)$;

2. $\phi_1(X) = \gamma(X)$.

In this way, the category of differential Lie crossed modules can be extended to a 2–category which is identified with the 2–category of strict 2–term L_∞ Lie algebras.

2.5 Examples of Lie 2–algebras

Below, we shall illustrate some simple but important examples of Lie 2–algebras.

1. Lie algebras

Every Lie algebra $\mathfrak{l}$ can be regarded as a strict 2–term L_∞ algebra, denoted by the same symbol. As a differential crossed module, $\mathfrak{l}$ is defined by the data

$(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$, where $\mathfrak{g} = \mathfrak{l}$, $\mathfrak{h} = 0$, $\tau : \mathfrak{h} \to \mathfrak{g}$ vanishes and $\mu : \mathfrak{g} \to \mathfrak{der}(\mathfrak{h})$ is trivial.

2. Inner derivation Lie 2–algebras

With any Lie algebra $\mathfrak{l}$, there is associated canonically a strict 2–term L_∞ algebra $\mathfrak{inn}(\mathfrak{l})$ defined as follows. As a differential crossed module, $\mathfrak{inn}(\mathfrak{l})$ is the quadruple of data $(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$, where $\mathfrak{g} = \mathfrak{l}$, $\mathfrak{h} = \mathfrak{l}$, $\tau : \mathfrak{h} \to \mathfrak{g}$ is the identity $\mathrm{id}_\mathfrak{l}$ and $\mu : \mathfrak{g} \to \mathfrak{der}(\mathfrak{h})$ is the adjoint action $\mathrm{ad}_\mathfrak{l}$ of $\mathfrak{l}$ on itself. $\mathfrak{inn}(\mathfrak{l})$ is called the *inner derivation Lie 2–algebra of* $\mathfrak{l}$.

3. Derivation Lie 2–algebras

The derivations of a Lie algebra $\mathfrak{l}$, $\mathfrak{der}(\mathfrak{l})$ form a Lie algebra and thus also a strict 2–term L_∞ algebra, by example 1. However, $\mathfrak{der}(\mathfrak{l})$ has a second strict 2–term L_∞ algebra structure defined as follows. Viewed again as a differential crossed module, $\mathfrak{der}(\mathfrak{l})$ is specified by the data $(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$, where $\mathfrak{g} = \mathfrak{der}(\mathfrak{l})$, $\mathfrak{h} = \mathfrak{l}$, $\tau : \mathfrak{h} \to \mathfrak{g}$ is the adjoint Lie algebra morphism $\mathrm{ad}_\mathfrak{l}$ and $\mu : \mathfrak{g} \to \mathfrak{der}(\mathfrak{h})$ is the identity $\mathrm{id}_{\mathfrak{der}(\mathfrak{l})}$. $\mathfrak{aut}(\mathfrak{l})$ is called the *derivation Lie 2–algebra of* $\mathfrak{l}$.

4. Central extension Lie 2–algebras

Consider a central extension of a Lie algebra $\mathfrak{l}$ by an Abelian Lie algebra $\mathfrak{a}$, that is a third Lie algebra $\mathfrak{e}$ fitting in a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{e} \longrightarrow \mathfrak{l} \longrightarrow 0, \tag{2.5.1}$$

with the image of $\mathfrak{a}$ contained in the center of $\mathfrak{e}$. With the extension, there is associated a canonical differential crossed module $(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$, hence a strict 2–term L_∞ algebra, as follows. $\mathfrak{g} = \mathfrak{l}$, $\mathfrak{h} = \mathfrak{e}$. $\tau : \mathfrak{h} \to \mathfrak{g}$ is the third morphism in the sequence (2.5.1). $\mu : \mathfrak{g} \to \mathfrak{der}(\mathfrak{h})$ is defined by choosing a linear mapping $\sigma : \mathfrak{l} \to \mathfrak{e}$ such that $\tau \circ \sigma = \mathrm{id}_\mathfrak{l}$ and setting $\mu(x)(X) = [\sigma(x), X]_\mathfrak{e}$. As σ is defined mod $\ker \tau$ which is contained in the center of $\mathfrak{e}$, μ is well–defined. The resulting Lie 2–algebra $\mathfrak{c}_\mathfrak{e}$ is called the *central extension Lie 2–algebra of* $\mathfrak{e}$.

Next we consider a few examples of non strict Lie 2–algebras.

5. Jacobiator Lie 2–algebras

A *pre–Lie algebra* is a vector space $\mathfrak{l}$ equipped with a linear map $[\cdot,\cdot]_{\mathfrak{l}}:\mathfrak{l}\wedge\mathfrak{l}\to\mathfrak{l}$. It is *not* assumed that $[\cdot,\cdot]_{\mathfrak{l}}$ satisfies the Jacobi identity.

Let $\mathfrak{l}$ be a pre–Lie algebra. Let $\mathfrak{v}_0=\mathfrak{l}$, $\mathfrak{v}_1=\mathfrak{l}$ and let $\partial:\mathfrak{v}_1\to\mathfrak{v}_0$ be the identity map $\mathrm{id}_{\mathfrak{l}}$. Further, let $[\cdot,\cdot]:\mathfrak{v}_0\wedge\mathfrak{v}_0\to\mathfrak{v}_0$ and $[\cdot,\cdot]:\mathfrak{v}_0\otimes\mathfrak{v}_1\to\mathfrak{v}_1$ be the bracket $[\cdot,\cdot]_{\mathfrak{l}}$ of $\mathfrak{l}$ and $[\cdot,\cdot,\cdot]:\mathfrak{v}_0\wedge\mathfrak{v}_0\wedge\mathfrak{v}_0\to\mathfrak{v}_1$ be defined by

$$[x,y,z]=[x,[y,z]_{\mathfrak{l}}]_{\mathfrak{l}}+[y,[z,x]_{\mathfrak{l}}]_{\mathfrak{l}}+[z,[x,y]_{\mathfrak{l}}]_{\mathfrak{l}}, \tag{2.5.2}$$

that is the Jacobiator of $[\cdot,\cdot]_{\mathfrak{l}}$. Then, $(\mathfrak{v}_0,\mathfrak{v}_1,\partial,[\cdot,\cdot],[\cdot,\cdot,\cdot])$ is a 2–term L_∞ algebra canonically associated with the pre–Lie algebra $\mathfrak{l}$, which we shall denote by $\mathfrak{j}_{\mathfrak{l}}$ and call the *Jacobiator Lie* 2*–algebra of* $\mathfrak{l}$. When $\mathfrak{l}$ is a Lie algebra, $\mathfrak{j}_{\mathfrak{l}}$ reduces to $\mathfrak{inn}(\mathfrak{l})$.

An important illustration of this is furnished by the imaginary octonions $\mathrm{Im}\,\mathbb{O}\simeq\mathbb{R}^7$. In this case, $[\cdot,\cdot]_{\mathrm{Im}\,\mathbb{O}}$ is the customary octonionic commutator

$$[x,y]_{\mathrm{Im}\,\mathbb{O}}=xy-yx, \tag{2.5.3}$$

$x,y\in\mathrm{Im}\,\mathbb{O}$. The Jacobiator algebra $\mathfrak{j}_{\mathrm{Im}\,\mathbb{O}}$ is therefore defined. Since octonionic multiplication is not associative, the associated 3 argument bracket is non trivial. Remarkably, the 2 argument and 3 argument brackets of $\mathfrak{j}_{\mathrm{Im}\,\mathbb{O}}$ are related in simple manner to the associative 3–form ϕ and coassociative 4–form ψ of $\mathrm{Im}\,\mathbb{O}$:

$$\phi(x,y,z)=-\frac{1}{2}\,\mathrm{Re}(x[y,z]), \tag{2.5.4}$$

$$\psi(x,y,z,w)=-\frac{1}{12}\,\mathrm{Re}(x[y,z,w]). \tag{2.5.5}$$

6. The string Lie 2*–algebra*

The *string Lie* 2*–algebra* $\mathfrak{string}_k(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra of compact type and $k\in\mathbb{R}$, is an important example of non strict Lie 2–algebra. For $\mathfrak{g}=\mathfrak{so}(n)$, it is relevant in string theory. As a 2–term L_∞ algebra, it can be presented as the set of data $(\mathfrak{v}_0,\mathfrak{v}_1,\partial,[\cdot,\cdot],[\cdot,\cdot,\cdot])$ defined as follows. $\mathfrak{v}_0=\mathfrak{g}$, $\mathfrak{v}_1=\mathbb{R}$. $\partial:\mathfrak{v}_1\to\mathfrak{v}_0$ vanishes. $[\cdot,\cdot]:\mathfrak{v}_0\wedge\mathfrak{v}_0\to\mathfrak{v}_0$ is the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$, $[\cdot,\cdot]:\mathfrak{v}_0\otimes\mathfrak{v}_1\to\mathfrak{v}_1$ vanishes and $[\cdot,\cdot,\cdot]:\mathfrak{v}_0\wedge\mathfrak{v}_0\wedge\mathfrak{v}_0\to\mathfrak{v}_1$ is defined by

$[x, y, z] = k\langle x, [y, z]_{\mathfrak{g}}\rangle$, where $\langle \cdot, \cdot \rangle$ is a suitably normalized invariant symmetric non singular bilinear form on $\mathfrak{g}$.

Another non strict example is provided by weak Courant–Dorfman algebras [49, 50], in particular by Courant algebroids [51].

2.6 2–groups

Algebraically, the finite counterpart of a Lie 2–algebra should be a 2–group. *Weak* or *coherent 2–groups*, or 2–groups for short, have been studied in depth in [20], which addresses various notions of 2–groups appeared in the literature giving a synthesis. A coherent 2–group is a category equipped with a multiplication, a unit and an inversion functor analogous to group operations but satisfying the associativity, unit and inverse law only up to coherent natural isomorphisms. As this definition already suggests, there are remarkable structural similarities between the theory of 2–groups and that of Lie 2–algebras. In particular, the classification theorem of Lie 2–algebras stated in subsect. 2.1 has a close 2–group analog. In [20], it is shown that there is a one–to–one correspondence between equivalence classes of 2–groups and isomorphism classes of the following data:

1. a group G,
2. an Abelian group H,
3. a homomorphism $\alpha : G \to \mathrm{Aut}(H)$,
4. an element $[a] \in H^3(G, H)$ of group cohomology of G with values in H.

The proof of the theorem also follows a similar course. The 2–group counterpart of a 2–term L_∞ algebra is a *special* 2–group, a 2–group which is skeletal, that is all isomorphic objects are equal, and such that the unit and inverse laws hold strictly. Every coherent 2–group V is equivalent to a special 2–group V_s, Isomorphism

classes of these latter can then be shown to be in one–to–one correspondence with the isomorphism classes of the above data.

A *Lie 2–group* is a 2–group, in which objects and morphisms are smooth manifolds and the multiplication, unit and inversion functors are smooth. In spite of the close formal similarities noticed above, a relationship between Lie 2–algebras and Lie 2–groups analogous to that existing between ordinary Lie algebras and Lie groups does not appear to exist. In fact, unlike what happens for groups, in general Lie 2–algebras do not straightforwardly integrate to Lie 2–groups. We illustrate this situation with the following classical example taken from ref. [52].

Suppose that $\mathfrak{g}$ is a simple Lie algebra of compact type. Let us look for a coherent Lie 2–group integrating the string Lie 2–algebra $\mathfrak{string}_k(\mathfrak{g})$ introduced at the end of subsect. 2.5. According to the Lie 2–algebras classification theorem of subsect. 2.1, $\mathfrak{string}_k(\mathfrak{g})$ corresponds to the Lie algebra $\mathfrak{g}$, the Abelian Lie algebra $\mathfrak{u}(1)$, the trivial homomorphism $\mathfrak{g} \to \mathfrak{der}(\mathfrak{u}(1))$ and the suitably normalized canonical $\mathfrak{u}(1)$–valued $\mathfrak{g}$–3–cocycle $j = \langle \cdot, [\cdot, \cdot]_{\mathfrak{g}} \rangle$. To build a 2–group G_k integrating $\mathfrak{string}(\mathfrak{g})_k$, we need somehow to map $H^3(\mathfrak{g}, \mathfrak{u}(1))$ into $H^3(G, \mathrm{U}(1))$. $H^3(\mathfrak{g}, \mathfrak{u}(1))$ contains a lattice Λ consisting of the integer multiples of $[j]$. Chern–Simons [53] and Cheeger–Simons [54] construct an inclusion $\iota : \Lambda \to H^3(G, \mathrm{U}(1))$. Thus, by the 2–group classification theorem recalled above, when $k \in \mathbb{Z}$, we can build a special 2–group G_k corresponding to the group G, the Abelian group $\mathrm{U}(1)$, the trivial homomorphism $G \to \mathrm{Aut}(\mathrm{U}(1))$ and the cohomology class $k\iota[j] \in H^3(G, \mathrm{U}(1))$. Unfortunately, for $k \neq 0$, G_k is not and cannot be a Lie 2–group, as there is no continuous representative of the cohomology class $k\iota[j]$ if G and $\mathrm{U}(1)$ are given the usual topology, except for the trivial case $k = 0$. More on this in subsect. 2.10.

The abstract categorical setting, in which 2–groups are defined, albeit very elegant and powerful from a mathematical perspective, makes it difficult to ma-

nipulate them in detailed field theoretic applications. For this and other reasons, in this paper, we shall base our formulation of semistrict higher gauge theory not directly on 2–group theory, but on Lie 2–algebra. However, when dealing with global issues in higher gauge theory, it is not possible to restrict oneself to the infinitesimal Lie 2–algebraic level. A 2–group structure is bound to emerge in a way or another. Our proposal is to make reference to the automorphism 2–group of the underlying Lie 2–algebra viewed as a 2–term L_∞ algebra. This is a 2–group of a special sort, called strict.

Strict 2–groups form a distinguished subclass of the class of 2–groups for which matters are much simpler. Strict Lie 2–groups integrate strict Lie 2–algebra algebras much as ordinary Lie groups integrate ordinary Lie algebras. Hence, they are of a special interest. There are other reasons why they are relevant for us. As already recalled, the automorphisms of a general 2–term L_∞ algebra form a strict 2–group. Further, the gauge transformation group of our version of higher gauge theory is an infinite dimensional strict 2–group. For these reasons, we shall concentrate exclusively on strict 2–groups in the next few sections.

2.7 Strict 2–groups

The theory of strict 2–groups is phrased most efficiently in the language of higher category theory. We shall restrict ourselves to providing only the basic definitions and properties. See ref. [20] for a comprehensive categorical treatment. Strict 2–groups are also intimately related to crossed modules and are so amenable to a more conventional Lie algebraic treatment.

A *strict 2–group* (in delooped form) consists of the following set of data:

1. a set of 1-cells V_1;

2. a composition law of 1–cells $\circ : V_1 \times V_1 \to V_1$;

3. a inversion law of 1–cells ${}^{-1_\circ} : V_1 \to V_1$;

4. a distinguished unit 1–cell $1 \in V_1$;

5. for each pair of 1–cells $a, b \in V_1$, a set of 2–cells $V_2(a, b)$;

6. for each quadruple of 1–cells $a, b, c, d \in V_1$, a horizontal composition law of 2–cells $\circ : V_2(a, c) \times V_2(b, d) \to V_2(b \circ a, d \circ c)$;

7. for each pair of 1–cells $a, b \in V_1$, a horizontal inversion law of 2–cells ${}^{-1_\circ}$: $V_2(a, b) \to V_2(a^{-1_\circ}, b^{-1_\circ})$;

8. for each triple of 1–cells $a, b, c \in V_1$, a vertical composition law of 2–cells $\cdot : V_2(a, b) \times V_2(b, c) \to V_2(a, c)$;

9. for each pair of 1–cells $a, b \in V_1$, a vertical inversion law of 2–cells ${}^{-1_\cdot}$: $V_2(a, b) \to V_2(b, a)$;

10. for each 1–cell a, a distinguished unit 2–cell $1_a \in V_2(a, a)$.

These are required to satisfy the following axioms.

$$(c \circ b) \circ a = c \circ (b \circ a), \tag{2.7.1a}$$

$$a^{-1_\circ} \circ a = a \circ a^{-1_\circ} = 1, \tag{2.7.1b}$$

$$a \circ 1 = 1 \circ a = a, \tag{2.7.1c}$$

$$(C \circ B) \circ A = C \circ (B \circ A), \tag{2.7.1d}$$

$$A^{-1_\circ} \circ A = A \circ A^{-1_\circ} = 1_1, \tag{2.7.1e}$$

$$A \circ 1_1 = 1_1 \circ A = A, \tag{2.7.1f}$$

$$(C \cdot B) \cdot A = C \cdot (B \cdot A), \tag{2.7.1g}$$

$$A^{-1_\cdot} \cdot A = 1_a, \qquad A \cdot A^{-1_\cdot} = 1_b, \tag{2.7.1h}$$

$$A \cdot 1_a = 1_b \cdot A = A, \tag{2.7.1i}$$

$$(D \cdot C) \circ (B \cdot A) = (D \circ B) \cdot (C \circ A). \tag{2.7.1j}$$

Here and in the following, $a, b, c, \cdots \in V_1$, $A, B, C, \cdots \in V_2$, where V_2 denotes the set of all 2-cells. For clarity, we often denote $A \in V_2(a, b)$ as $A : a \Rightarrow b$. All identities involving the vertical composition and inversion hold whenever defined. Relation (2.7.1j) is called interchange law. In the following, we shall denote a 2–group such as the above as V or (V_1, V_2) or $(V_1, V_2, \circ, {}^{-1\circ}, \cdot, {}^{-1\cdot}, 1_{-})$ to emphasize the underlying structure.

If $(V_1, V_2, \circ, {}^{-1\circ}, \cdot, {}^{-1\cdot}, 1_{-})$ is a strict 2–group, then $(V_1, \circ, {}^{-1\circ}, 1)$ is an ordinary group and $(V_1, V_2, \cdot, {}^{-1\cdot}, 1_{-})$ is a groupoid. Viewing this as a category V, $\circ : V \times V \to V$ and ${}^{-1\circ} : V \to V$ are both functors. Indeed, V is a strict monoidal category in which every morphism is invertible and every object has a strict inverse. V can also be viewed as a one–object strict 2–category in which all 1–morphisms are invertible and all 2–morphisms are both horizontal and vertical invertible, that is a one–object strict 2–groupoid.

A *crossed module* [55] consists in the following elements.

1. a pair of groups G, H;

2. a group morphism $t : H \to G$;

3. a group morphism $m : G \to \mathrm{Aut}(H)$, where $\mathrm{Aut}(H)$ is the group of automorphisms of H.

Further, the following conditions are met.

$$t(m(a)(A)) = at(A)a^{-1}, \tag{2.7.2a}$$

$$m(t(A))(B) = ABA^{-1}, \tag{2.7.2b}$$

where here and in the following $a, b, c, \cdots \in G$, $A, B, C, \cdots \in H$. We shall denote a crossed module such as this by (G, H) or (G, H, t, m) to explicitly indicate its underlying structure.

There exists a one–to–one correspondence between strict 2–groups and crossed

modules [56]. With any crossed module (G, H), there is associated a strict 2–group V as follows.

1. $V_1 = G$;
2. $b \circ a = ba$;
3. $a^{-1\circ} = a^{-1}$;
4. $1 = 1_G$;
5. $V_2(a, b)$ is the set of pairs $(a, A) \in G \times H$ such that $b = t(A)a$;
6. $(b, B) \circ (a, A) = (ba, Bm(b)(A))$;
7. $(a, A)^{-1\circ} = (a^{-1}, m(a^{-1})(A^{-1}))$;
8. for composable (a, A), (b, B), $(b, B) \cdot (a, A) = (a, BA)$;
9. $(a, A)^{-1\cdot} = (t(A)a, A^{-1})$;
10. $1_a = (a, 1_H)$.

Conversely, with any strict 2–group V there is associated a crossed module (G, H), as follows.

1. $G = V_1$;
2. $ba = b \circ a$;
3. $a^{-1} = a^{-1\circ}$;
4. $1_G = 1$;
5. H is the set of all 2–cells of the form $A : 1 \Rightarrow a$ for some a;
6. $BA = B \circ A$;

7. $A^{-1} = A^{-1_\circ}$;

8. $1_H = 1_1$;

9. $t(A) = a$ if $A : 1 \Rightarrow a$.

10. $m(a)(A) = 1_a \circ A \circ 1_{a^{-1_\circ}}$.

A strict Lie 2–group is a strict 2–group $(V_1, V_2, \circ, {}^{-1_\circ}, \cdot, {}^{-1_\cdot}, 1_{-})$ such that V_1, V_2 are smooth manifolds and $\circ, {}^{-1_\circ}, \cdot, {}^{-1_\cdot}, 1_{-}$ are smooth mappings. Similarly, a Lie crossed module is a crossed module (G, H, t, m) such that G, H are Lie groups and t, m are smooth mappings.

Let V, V' be strict 2–groups. A *strict 2–group 1–morphism* from V to V' is a pair of

1. a mapping $\theta_1 : V_1 \to V'_1$,

2. for any two 1–cells $a, b \in V_1$, a mapping $\theta_2 : V_2(a, b) \to V'_2(\theta_1(a), \theta_1(b))$

preserving the 2–group structure:

$$\theta_1(b \circ a) = \theta_1(b) \circ \theta_1(a), \tag{2.7.3a}$$

$$\theta_1(a^{-1_\circ}) = \theta_1(a)^{-1_\circ}, \tag{2.7.3b}$$

$$\theta_1(1) = 1', \tag{2.7.3c}$$

$$\theta_2(B \circ A) = \theta_2(B) \circ \theta_2(A), \tag{2.7.3d}$$

$$\theta_2(A^{-1_\circ}) = \theta_2(A)^{-1_\circ}, \tag{2.7.3e}$$

$$\theta_2(B \cdot A) = \theta_2(B) \cdot \theta_2(A), \tag{2.7.3f}$$

$$\theta_2(A^{-1_\cdot}) = \theta_2(A)^{-1_\cdot}, \tag{2.7.3g}$$

$$\theta_2(1_a) = 1'_{\theta_1(a)}. \tag{2.7.3h}$$

We shall denote such a 1–morphism as θ or, more explicitly, as (θ_1, θ_2). We shall also write $\theta : V \to V'$ to emphasize the source and target 2–groups.

If $\theta : V \to V'$ is a strict 2–group 1–morphism, then $\theta_1 : V_1 \to V'_1$ is a group morphism and $\theta : (V_1, V_2) \to (V'_1, V'_2)$ is a groupoid morphism. $\theta : V \to V'$ can also be viewed as a 2–functor of the 2–categories V, V'.

It is also possible to introduce the notion of 2–morphism from a 1–morphism to another. For any two strict 2–group 1–morphisms $\theta, \upsilon : V \to V'$, a strict 2–group 2–morphism from θ to υ consists of a full set of data of the form

1. for any two $a \in V_1$, an element $\Theta(a) \in V'_2(\theta_1(a), \upsilon_1(a))$

such that the following relations are satisfied

$$\Theta(b \circ a) = \Theta(b) \circ \Theta(a), \tag{2.7.4a}$$

$$\Theta(a^{-1\circ}) = \Theta(a)^{-1\circ}, \tag{2.7.4b}$$

$$\Theta(1) = 1'_1, \tag{2.7.4c}$$

$$\Theta(b) \cdot \theta_2(A) = \upsilon_2(A) \cdot \Theta(a), \tag{2.7.4d}$$

where $A : a \Rightarrow b$. We shall denote a morphism such as this as Θ or more explicitly as $\Theta : \theta \Rightarrow \upsilon$.

If $\theta, \upsilon : V \to V'$ are 1–morphisms and $\Theta : \theta \Rightarrow \upsilon$ is a 2–morphism, then Θ is a pseudonatural transformation of the 2–functors θ, υ.

Next, having in mind a 2–categorical structure, we shall define a composition law and a unit for 1–morphisms and horizontal and vertical composition laws and units for 2–morphisms.

The composition of two strict 2–group 1–morphisms $\theta : V \to V'$, $\upsilon : V' \to V''$ is the 1–morphism $\upsilon \circ \theta : V \to V''$, defined componentwise by

$$(\upsilon \circ \theta)_1(a) = \upsilon_1(\theta_1(a)), \tag{2.7.5a}$$

$$(\upsilon \circ \theta)_2(A) = \upsilon_2(\theta_2(A)). \tag{2.7.5b}$$

For any strict 2–group V, the identity of V is the strict 2–group 1–morphism

$\mathrm{id}_V : V \to V$ defined componentwise by

$$\mathrm{id}_{V1}(a) = a, \tag{2.7.6a}$$

$$\mathrm{id}_{V2}(A) = A. \tag{2.7.6b}$$

The horizontal composition of two strict 2–group 2–morphisms $\Lambda : \phi \Rightarrow \psi$, $\Theta : \lambda \Rightarrow \mu$, with $\phi, \psi : V \to V'$, $\lambda, \mu : V' \to V''$ strict 2–group 1–morphisms, is the 2–morphism $\Theta \circ \Lambda : \lambda \circ \phi \Rightarrow \mu \circ \psi$ defined by

$$\Theta \circ \Lambda(a) = \Theta(\psi_1(a)) \cdot \lambda_2(\Lambda(a)) = \mu_2(\Lambda(a)) \cdot \Theta(\phi_1(a)). \tag{2.7.7}$$

The vertical composition of two strict 2–group 2–morphisms $\Pi : \lambda \Rightarrow \mu$, $\Lambda : \mu \Rightarrow \nu$, with $\lambda, \mu, \nu : V \to V'$ strict 2–group 1–morphisms, is the 2–morphism $\Lambda \cdot \Pi : \lambda \Rightarrow \nu$ defined by

$$\Lambda \cdot \Pi(a) = \Lambda(a) \cdot \Pi(a). \tag{2.7.8}$$

Finally, for any strict 2–group 1–morphism $\theta : V \to V'$, the identity of θ is the 2–group 2–morphism $\mathrm{Id}_\phi : \theta \Rightarrow \theta$ given by

$$\mathrm{Id}_\theta(a) = 1'_{\theta_1(a)}. \tag{2.7.9}$$

The composition of 1–morphisms, the unit of an L_∞ algebra, the horizontal and vertical composition of 2–morphisms and the unit of a 1–morphism satisfy the following basic relations

$$(\nu \circ \mu) \circ \lambda = \nu \circ (\mu \circ \lambda), \tag{2.7.10a}$$

$$\lambda \circ \mathrm{id}_V = \mathrm{id}_{V'} \circ \lambda = \lambda, \tag{2.7.10b}$$

$$(\Pi \circ \Lambda) \circ \Theta = \Pi \circ (\Lambda \circ \Theta), \tag{2.7.10c}$$

$$\Theta \circ \mathrm{Id}_{\mathrm{id}_V} = \mathrm{Id}_{\mathrm{id}_{V'}} \circ \Theta = \Theta, \tag{2.7.10d}$$

$$(\Pi \cdot \Lambda) \cdot \Theta = \Pi \cdot (\Lambda \cdot \Theta), \tag{2.7.10e}$$

$$\Theta \cdot \mathrm{Id}_\lambda = \mathrm{Id}_\mu \cdot \Theta = \Theta, \tag{2.7.10f}$$

$$(\Xi \cdot \Pi) \circ (\Lambda \cdot \Theta) = (\Xi \circ \Lambda) \cdot (\Pi \circ \Theta), \tag{2.7.10g}$$

holding whenever the various instances of morphism composition are defined. (2.7.10) are precisely the relations which render the class of strict 2–groups a (strict) 2–category.

Let (G, H), (G', H') be crossed modules. A *crossed module morphism* from (G, H) to (G', H') is a pair of

1. a group morphism $\rho : G \to G'$,

2. a group morphism $\sigma : H \to H'$

preserving the crossed module relations,

$$\rho(t(A)) = t'(\sigma(A)), \tag{2.7.11a}$$

$$\sigma(m(a)(A)) = m'(\rho(a))(\sigma(A)). \tag{2.7.11b}$$

We shall denote a crossed module morphism like the above as (ρ, σ) or $(\rho, \sigma) : (G, H) \to (G', H')$.

With the morphism structure just defined, strict 2–groups form a category.

There is a obvious one–to–one correspondence between 2–group 1–morphism $\theta : V \to V'$ and crossed module morphism $(\rho, \sigma) : (G, H) \to (G', H')$, obtained by viewing the strict 2–groups V, V' as the crossed modules (G, H), (G', H'), as indicated above.

1. $\theta_1(a) = \rho(a)$;

2. $\theta_2(a, A) = (\rho(a), \sigma(A))$.

In this way, the category of crossed modules can be extended to a 2–category which is identified with the 2–category of strict 2–groups in the way explained in detail above.

If V, V' are strict Lie 2–groups, a strict Lie 2–group 1–morphism $\theta : V \to V'$ is a strict 2–group 1 morphism such that θ_1, θ_2 are both smooth. One can similarly define strict Lie 2–group 1–morphism $\Theta : \lambda \Rightarrow \mu$. Similarly, if (G, H), (G', H') are Lie crossed modules, a Lie crossed module morphism $(\rho, \sigma) : (G, H) \to (G', H')$ is a crossed module morphism such that ρ, σ are both smooth.

2.8 Strict Lie 2–groups and strict Lie 2–algebras

Much as with any Lie group G, there is associated a Lie algebra $\mathfrak{g}$, with any strict Lie 2–group V, there is associated a strict Lie 2–algebra $\mathfrak{v}$. Showing this is straightforward, if one sees the former as a Lie crossed module (G, H) and the latter as a differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$.

Let (G, H) be a Lie crossed module. With the group morphism $t : H \to G$, there is associated the Lie algebra morphism $\dot{t} : \mathfrak{h} \to \mathfrak{g}$ defined by

$$\dot{t}(X) = \frac{dt(C(s))}{ds}\Big|_{s=0}, \tag{2.8.1}$$

with $C(s)$ is any curve in H such that $C(s)\big|_{s=0} = 1_H$ and $dC(s)/ds\big|_{s=0} = X$. Likewise, with the group morphism $m : G \to \mathrm{Aut}(H)$ there is associated the Lie algebra morphism $\widehat{m} : \mathfrak{g} \to \mathfrak{der}(\mathfrak{h})$ as follows. For $a \in G$, let $\dot{m}(a) : \mathfrak{h} \to \mathfrak{h}$ be the vector space morphism given by

$$\dot{m}(a)(X) = \frac{dm(a)(C(s))}{ds}\Big|_{s=0}, \tag{2.8.2}$$

where $C(s)$ is any curve in H such that $C(s)\big|_{s=0} = 1_H$ and $dC(s)/ds\big|_{s=0} = X$. Then, viewing $\widehat{m} : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$, we have

$$\widehat{m}(x)(X) = \frac{d\dot{m}(c(u)))(X)}{du}\Big|_{u=0}, \tag{2.8.3}$$

where $c(u)$ is any curve in G such that $c(u)\big|_{u=0} = 1_G$ and $dc(u)/du\big|_{u=0} = x$.

We can now attach to a Lie crossed module (G, H, t, m) canonically a differential Lie crossed module $(\mathfrak{g}, \mathfrak{h}, \tau, \mu)$ as follows.

1. $\mathfrak{g} = \mathrm{Lie}\, G$;

2. $\mathfrak{h} = \mathrm{Lie}\, H$.

3. $\tau = \dot{t}$;

4. $\mu = \widehat{m}$.

The resulting correspondence can be phrased in the language of strict 2–groups and strict 2–term L_∞ algebras using the results of subsects. 2.4, 2.7.

2.9 The strict Lie 2–group of 2–term L_∞ algebra automorphisms

The notion of 2–term L_∞ algebra automorphism is central in the theory of 2–term L_∞ algebras. In this section, we shall show that the automorphisms of a 2–term L_∞ algebra $\mathfrak{v}$ form a strict 2–group $\mathrm{Aut}(\mathfrak{v})$. See again [21].

A 1–*automorphism* of $\mathfrak{v}$ is 2–term L_∞ algebra 1–morphism $\phi : \mathfrak{v} \to \mathfrak{v}$ such that $\phi_0 : \mathfrak{v}_0 \to \mathfrak{v}_0$ and $\phi_1 : \mathfrak{v}_1 \to \mathfrak{v}_1$ (cf. subsect. 2.3). We shall denote the set of all 1–automorphisms of $\mathfrak{v}$ by $\mathrm{Aut}_1(\mathfrak{v})$.

For any two 1–automorphisms ϕ, ψ, a 2–*automorphism* from ϕ to ψ is just a 2–term L_∞ algebra 2–morphism $\varPhi : \phi \Rightarrow \psi$ (cf. subsect. 2.3). We shall denote the set of all 2–automorphisms $\varPhi : \phi \Rightarrow \psi$ by $\mathrm{Aut}_2(\mathfrak{v})(\phi, \psi)$ and the set of all 2–automorphisms $\varPhi$ by $\mathrm{Aut}_2(\mathfrak{v})$.

A 1–automorphism $\phi \in \mathrm{Aut}_1(\mathfrak{v})$ is invertible as a 1–morphism $\phi : \mathfrak{v} \to \mathfrak{v}$, that is there exists a 1–morphism $\phi^{-1\circ} : \mathfrak{v} \to \mathfrak{v}$ such that $\phi^{-1\circ} \circ \phi = \phi \circ \phi^{-1\circ} = \mathrm{id}_{\mathfrak{v}}$. Explicitly, writing $\phi^{-1\circ} = (\phi^{-1\circ}{}_0, \phi^{-1\circ}{}_1, \phi^{-1\circ}{}_2)$, we have

$$\phi^{-1\circ}{}_0(x) = \phi_0{}^{-1}(x), \tag{2.9.1a}$$

$$\phi^{-1\circ}{}_1(X) = \phi_1{}^{-1}(X), \tag{2.9.1b}$$

$$\phi^{-1\circ}{}_2(x, y) = -\phi_1{}^{-1}\phi_2(\phi_0{}^{-1}(x), \phi_0{}^{-1}(y)). \tag{2.9.1c}$$

A 2–automorphism $\varPhi \in \mathrm{Aut}_2(\mathfrak{v})$ is both horizontally and vertically invertible as a 2–morphism $\varPhi : \phi \Rightarrow \psi$, that is there are a 2–morphism $\varPhi^{-1\circ} : \phi^{-1\circ} \Rightarrow \psi^{-1\circ}$ such that $\varPhi^{-1\circ} \circ \varPhi = \varPhi \circ \varPhi^{-1\circ} = \mathrm{Id}_{\mathrm{id}}$, and a 2–morphism $\varPi^{-1\cdot} : \psi \Rightarrow \phi$ such that $\varPhi^{-1\cdot} \cdot \varPhi = \mathrm{Id}_\phi$, $\varPhi \cdot \varPhi^{-1\cdot} = \mathrm{Id}_\psi$, Explicitly, we have

$$\varPhi^{-1\circ}(x) = -{\phi_1}^{-1}\varPhi{\psi_0}^{-1}(x) = -{\psi_1}^{-1}\varPhi{\phi_0}^{-1}(x), \tag{2.9.2}$$

$$\varPhi^{-1\cdot}(x) = -\varPhi(x). \tag{2.9.3}$$

$\mathrm{Aut}_1(\mathfrak{v})$ and $\mathrm{Aut}_2(\mathfrak{v})$ are subsets of the set 2–term L_∞ algebra 1–morphism and 2–morphisms, respectively. $\mathrm{Aut}_1(\mathfrak{v})$ is so endowed with a composition and an inversion law and a unit. $\mathrm{Aut}_2(\mathfrak{v})$ is similarly endowed with horizontal and vertical composition and inversion laws and units. It is straightforward though lengthy to check that these composition, inversion and unit structures satisfy the axioms (2.7.1) rendering $(\mathrm{Aut}_1(\mathfrak{v}), \mathrm{Aut}_2(\mathfrak{v}))$ a strict 2–group, as announced.

$\mathrm{Aut}(\mathfrak{v})$ is actually a strict Lie 2–group. Its associated strict 2–term L_∞ Lie algebra $\mathfrak{aut}(\mathfrak{v})$ is is described as follows.

An element of $\mathfrak{aut}_0(\mathfrak{v})$ consists of three mappings.

1. a vector space morphism $\alpha_0 : \mathfrak{v}_0 \to \mathfrak{v}_0$;

2. a vector space morphism $\alpha_1 : \mathfrak{v}_1 \to \mathfrak{v}_1$;

3. a vector space morphism $\alpha_2 : \mathfrak{v}_0 \wedge \mathfrak{v}_0 \to \mathfrak{v}_1$.

These must satisfy the following relations:

$$\alpha_0(\partial X) - \partial\alpha_1(X) = 0, \tag{2.9.4a}$$

$$\alpha_0([x, y]) - [\alpha_0(x), y] - [x, \alpha_0(y)] - \partial\alpha_2(x, y) = 0, \tag{2.9.4b}$$

$$\alpha_1([x, X]) - [\alpha_0(x), X] - [x, \alpha_1(X)] - \alpha_2(x, \partial X) = 0, \tag{2.9.4c}$$

$$[x, \alpha_2(y, z)] + [y, \alpha_2(z, x)] + [z, \alpha_2(x, y)] + \alpha_2(x, [y, z]) \tag{2.9.4d}$$

$$+ \alpha_2(y,[z,x]) + \alpha_2(z,[x,y]) - \alpha_1([x,y,z]) + [x,y,\alpha_0(z)]$$

$$+ [y,z,\alpha_0(x)] + [z,x,\alpha_0(y)] = 0.$$

An element of $\mathfrak{aut}_1(\mathfrak{v})$ consists of a single mapping mapping.

1. a vector space morphism $\mathit{\Gamma} : \mathfrak{v}_0 \to \mathfrak{v}_1$.

No restrictions are imposed on it.

The boundary map and the brackets of $\mathfrak{aut}(\mathfrak{v})$ are given by the expressions

$$\partial_{\mathrm{aut}} \mathit{\Gamma}_0(x) = -\partial \mathit{\Gamma}(x), \tag{2.9.5a}$$

$$\partial_{\mathrm{aut}} \mathit{\Gamma}_1(X) = -\mathit{\Gamma}(\partial X), \tag{2.9.5b}$$

$$\partial_{\mathrm{aut}} \mathit{\Gamma}_2(x,y) = [x,\mathit{\Gamma}(y)] - [y,\mathit{\Gamma}(x)] - \mathit{\Gamma}([x,y]), \tag{2.9.5c}$$

$$[\alpha,\beta]_{\mathrm{aut}0}(x) = \alpha_0\beta_0(x) - \beta_0\alpha_0(x), \tag{2.9.5d}$$

$$[\alpha,\beta]_{\mathrm{aut}1}(X) = \alpha_1\beta_1(X) - \beta_1\alpha_1(X), \tag{2.9.5e}$$

$$\begin{aligned}[\alpha,\beta]_{\mathrm{aut}2}(x,y) = \alpha_1\beta_2(x,y) + \alpha_2(\beta_0(x),y) + \alpha_2(x,\beta_0(y)) \\ - \beta_1\alpha_2(x,y) - \beta_2(\alpha_0(x),y) - \beta_2(x,\alpha_0(y)),\end{aligned} \tag{2.9.5f}$$

$$[\alpha,\mathit{\Gamma}]_{\mathrm{aut}}(x) = \alpha_1\mathit{\Gamma}(x) - \mathit{\Gamma}\alpha_0(x), \tag{2.9.5g}$$

$$[\alpha,\beta,\gamma]_{\mathrm{aut}}(x) = 0. \tag{2.9.5h}$$

Relations (2.9.4) ensure that the basic relations (2.2.1) are satisfied.

For a Lie group G with Lie algebra $\mathfrak{g}$, there exists is a canonical group morphism of G into $\mathrm{Aut}(\mathfrak{g})$ stemming from the Lie group structure of G, defining the adjoint representation of G. We are now going to see that this property generalizes to a strict Lie 2–group V with strict Lie 2–algebra $\mathfrak{v}$ by constructing a canonical 2–group morphism of V into $\mathrm{Aut}(\mathfrak{v})$. Though we have not defined the notion of representation of a strict 2–group, we can consider rightfully this morphism to be the strict 2–group generalization of the adjoint representation of an

ordinary Lie group. To this end, we view V as a Lie crossed module (G, H) and $\mathfrak{v}$ as the corresponding differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$. With any $a \in V_1$, we associate a 1–automorphism of $\mathfrak{v}$ defined by

$$\phi_{a0}(x) = axa^{-1}, \tag{2.9.6a}$$

$$\phi_{a1}(X) = \dot{m}(a)(X), \tag{2.9.6b}$$

$$\phi_{a2}(x, y) = 0. \tag{2.9.6c}$$

Further, with any $(a, A) \in V_2(a, b)$ with $b = t(A)a$, we associate a 2–automorphism $\Phi_{a,A} : \phi_a \Rightarrow \phi_b$ of $\mathfrak{v}$ defined by

$$\Phi_{a,A}(x) = Q(axa^{-1}, A), \tag{2.9.7}$$

where, for $A \in H$, $Q(\cdot, A) : \mathfrak{g} \to \mathfrak{h}$ is the vector space morphism defined by

$$Q(x, A) = \frac{d}{du} m(c(u))(A)A^{-1}\Big|_{u=0}, \tag{2.9.8}$$

with $c(u)$ is a curve in G such that $c(0) = 1_G$ and $dc(u)/du\big|_{u=0} = x$ [3]. Now, it is straightforward to verify that the mappings $a \to \phi_a$ and $(a, A) \to \Phi_{a,A}$ define a strict 2–group 1–morphism from V to $\mathrm{Aut}(\mathfrak{v})$ as desired.

2.10 Examples of Lie 2–groups

Below, we shall illustrate some simple but important examples of 2–groups.

[3] Q has the following properties, which turn out to be relevant,

$$Q([x, y], A) + [Q(x, A), Q(y, A)] - [x, Q(y, A)] + [y, Q(x, A)] = 0, \tag{2.9.9a}$$

$$Q(x, AB) = Q(x, A) + AQ(x, B)A^{-1}, \tag{2.9.9b}$$

$$Q(axa^{-1}, A) = \dot{m}(a)(Q(x, m(a^{-1})(A))). \tag{2.9.9c}$$

By relation (2.9.9a), for fixed A, the mapping $x \to x - Q(x, A)$ is a Lie algebra morphism of $\mathfrak{g}$ into the semidirect sum $\mathfrak{g} \in\!\!\!\!\!\! + \mathfrak{h}$. By relation (2.9.9b), for fixed x, $A \to Q(x, A)$ is a $\mathfrak{h}$–valued H 1–cocycle. Relation (2.9.9c) implies that $x \to Q(x, 1_H)$ is G–equivariant

1. Lie groups

Every Lie group L can be regarded as a strict Lie 2–group, denoted by the same symbol. As a Lie crossed module, L is defined by the data (G, H, t, m), where $G = L$, $H = 1$, $t : H \to G$ vanishes and $m : G \to \mathrm{Aut}(H)$ is trivial. The Lie 2–algebra of L as a Lie 2–group is the Lie algebra $\mathfrak{l}$ of L as a Lie group regarded as Lie 2–algebra (cf. sect. 2.5).

2. Inner automorphism Lie 2–groups

With any Lie group L, there is associated canonically a strict Lie 2–group $\mathrm{Inn}(L)$ defined as follows. As a Lie crossed module, $\mathrm{Inn}(L)$ is the quadruple of data (G, H, t, m), where $G = L$, $H = L$, $t : H \to G$ is the identity id_L and $m : G \to \mathrm{Aut}(H)$ is the adjoint action Ad_L of L on itself. $\mathrm{Inn}(L)$ is called *inner automorphism Lie 2–group of L*. The Lie 2–algebra of $\mathrm{Inn}(L)$ is the inner derivation Lie 2–algebra $\mathfrak{inn}(\mathfrak{l})$ of the Lie algebra $\mathfrak{l}$ of L (cf. sect. 2.5).

3. Automorphism Lie 2–groups

The automorphisms of a Lie group L, $\mathrm{Aut}(L)$ form a Lie group and thus also a strict 2–group, by example 1. However, $\mathrm{Aut}(L)$ has a second strict 2–group structure defined as follows. Viewed again as a Lie crossed module, $\mathrm{Aut}(L)$ is specified by the data (G, H, t, m), where $G = \mathrm{Aut}(L)$, $H = L$, $t : H \to G$ is the adjoint Lie group morphism Ad_L and $m : G \to \mathrm{Aut}(H)$ is the identity $\mathrm{id}_{\mathrm{Aut}(L)}$. $\mathrm{Aut}(L)$ is called *automorphism Lie 2–group of L*. The Lie 2–algebra of $\mathrm{Aut}(L)$ is the derivation Lie 2–algebra $\mathfrak{Der}(\mathfrak{l})$ of the Lie algebra $\mathfrak{l}$ of L (cf. sect. 2.5).

4. Central extension Lie 2–groups

Consider a central extension of a Lie group L by an Abelian Lie group A, that is a third Lie group E fitting in a short exact sequence of Lie groups

$$1 \longrightarrow A \longrightarrow E \longrightarrow L \longrightarrow 1, \tag{2.10.1}$$

with the image of A contained in the center of E. With the extension, there is associated a canonical Lie crossed module (G, H, t, m), hence a strict 2–group,

as follows. $G = L$, $H = E$. $t : H \to G$ is the third morphism in the sequence (2.10.1). $m : G \to \mathrm{Aut}(H)$ is defined by choosing a linear mapping $s : L \to E$ such that $t \circ s = \mathrm{id}_L$ and setting $m(a)(A) = s(a)As(a)^{-1}$. As s is defined mod $\ker t$ which is contained in the center of E, m is well–defined. The resulting strict Lie 2–group C_E is called the *central extension Lie 2–group of E*. The Lie algebra $\mathfrak{e}$ of E is a central extension of the Lie algebra $\mathfrak{l}$ of L by the Abelian Lie algebra $\mathfrak{a}$ of A. The Lie 2–algebra of C_E is the central extension Lie 2–algebra $\mathfrak{c}_{\mathfrak{e}}$ of $\mathfrak{e}$ (cf. sect. 2.5, ex. 4).

Next, we consider a few non strict examples.

5. Non associative structures and Moufang loops

In sect. 2.5, we have introduced the Jacobiator Lie 2–algebra $\mathfrak{j}_{\mathfrak{l}}$ associated to a Lie prealgebra $\mathfrak{l}$, which is generally non strict. It is natural to wonder whether there are higher group like structures which have such Lie 2–algebras as infinitesimal counterparts. When $\mathfrak{l}$ is a Lie algebra, $\mathfrak{j}_{\mathfrak{l}} = \mathfrak{inn}(\mathfrak{l})$, which is the Lie 2–algebra of the inner automorphism 2–group $\mathrm{Inn}(L)$, where L is a Lie group integrating L. When $\mathfrak{l}$ is not a Lie algebra, things are not so clear. To the best of our knowledge, not much is known about this matter. Consider for instance the Jacobiator Lie 2–algebra of the imaginary octonions $\mathrm{Im}\,\mathbb{O}$. $\mathrm{Im}\,\mathbb{O}$ is the tangent space at 1 of the 7–sphere of the unit norm octonions $\mathrm{U}(\mathbb{O})$. As octonions are not associative, $\mathrm{U}(\mathbb{O})$ is neither a group nor a strict 2–group under octonionic multiplication. But $\mathrm{U}(\mathbb{O})$ does not constitute even a coherent 2–group. It has rather the structure of a *Moufang loop* [57]. A loop is a set S equipped with a generally non associative binary operation $(x, y) \to xy$ with the following properties:

1. for any $x, y \in S$ there exists $u, v \in S$ satisfying the relations $ux = y$, $xv = y$;

2. there distinguished element $1 \in S$ such that $x1 = 1x = x$ for $x \in S$.

A Moufang Loop is a loop where the Moufang identity $(zx)(yz) = (z(xy))z =$

$z((xy)z)$ is satisfied identically for $x, y, z \in S$. The stronger condition $w(x(yz)) = ((wx)y)z$ does not hold, as no associator is available in a Moufang loop.

6. The string 2–group

Let G be a simply-connected, connected, compact simple Lie group G. The loop group ΩG of G is the infinite dimensional Lie group of the smooth loops of G based at 1_G equipped with pointwise multiplication. In ref. [58], Pressley and Segal showed that, for each integer k, ΩG has a central extension

$$1 \longrightarrow U(1) \longrightarrow \widehat{\Omega}_k G \longrightarrow \Omega G \longrightarrow 1. \tag{2.10.2}$$

k is called the level of the extension. The extensions of different levels are inequivalent.

Proceeding as explained in ex. 4, one builds the infinite dimensional central extension Lie 2–group of $\widehat{\Omega}_k G$, the *level k loop 2–group* $L_k(G)$ of G. In ref. [52], Baez et al. showed that $L_k(G)$ fits into an exact sequence

$$1 \longrightarrow L_k(G) \longrightarrow \mathrm{String}_k(G) \longrightarrow G \longrightarrow 1, \tag{2.10.3}$$

of strict Lie 2–groups. The middle term of the sequence is an infinite dimensional strict Lie 2–group, the *level k string 2–group* $\mathrm{String}_k(G)$ of G. The infinite-dimensional strict Lie 2–algebra of $\mathrm{String}_k(G)$ is equivalent to the string Lie 2–algebra $\mathfrak{string}_k(\mathfrak{g})$. Note that $\mathrm{String}_k(G)$ is defined only for integer k while $\mathfrak{string}_k(\mathfrak{g})$ is for any real k. The relationship just described between $\mathrm{String}_k(G)$ and $\mathfrak{string}_k(\mathfrak{g})$, so, holds only when k is integer.

In [52] Baez et al. did not integrate the string Lie 2–algebra $\mathfrak{string}_k(\mathfrak{g})$. They only constructed $\mathrm{String}_k(G)$ and showed that its infinite dimensional Lie 2-algebra is equivalent to $\mathfrak{string}_k(\mathfrak{g})$. In [59], Henriques worked out a procedure to integrate $\mathfrak{string}_1(\mathfrak{g})$. The resulting model of $\mathrm{String}_1(G)$ is again an infinite dimensional Lie 2–group.

In ref. [60], Schommer-Pries managed to produce a finite dimensional model of the string 2–group $\mathrm{String}_1(G)$. The price paid for this is the need for a notion of

Lie 2–group weaker than the customary one of coherent Lie 2–group. Schommer-Pries' smooth 2–groups are 2–group objects in a bicategory of Lie groupoids, left-principal bibundles, and bibundle maps rather than the 2–category of Lie groupoids, smooth functors and smooth natural transformations.

The above discussion shows once more that the relation between coherent Lie 2–groups and Lie 2–algebras is rather subtle. While any strict Lie 2–algebra always integrates to a strict Lie 2–group, not so Lie 2–algebras.

3 Semistrict higher gauge theory

We can now start illustrating our formulation of *semistrict algebra gauge theory*. For an alternative approaches to higher gauge theory see ref. [19].

As we recalled in the introduction, the basic geometric structure of ordinary gauge theory is a principal bundle $P(M, G)$ on a manifold M with structure Lie group G. The Lie algebra of G, $\mathfrak{g}$, is a derived secondary object. A formulation on the same lines of semistrict higher gauge theory would require a principal 2–bundle $P(M, V)$ on a 2–manifold M with a structure Lie 2–group V of the appropriate type along the lines of [32, 33]. Again, the Lie 2–algebra of V, $\mathfrak{v}$, would be a derived secondary object. This type of approach, while the most powerful in theory, is likely to be very difficult to implement in practice. The relation between Lie 2–groups and Lie 2-algebras is a sticky matter beyond the strict case. If we demand as we do that $\mathfrak{v}$ be a semistrict Lie 2–algebra then V may be something more general than a mere coherent Lie 2–group, as the case of the string Lie 2–algebra shows. In particular, it may be infinite dimensional. Though general techniques to cope with these difficulties have been developed recently [35, 36], their abstractness makes it difficult their application to detailed calculations of the type presented in the second half of this paper. For this reason, we have opted for a more conventional and conservative approach, which exploits as much as possible the computational effectiveness of 2–term L_∞ algebras, as we illustrate next.

In ordinary gauge theory, the fields are $\mathfrak{g}$–valued and their global properties are controlled by a set of Čech gluing data hinging on an $\mathrm{Aut}(\mathfrak{g})$–valued cocycle acting by gauge transformations and obeying certain coherence conditions. The theory, so, can be formulated to a substantial extent relying on the structure Lie algebra $\mathfrak{g}$ only and its topological features are encoded in those data. In the same way, in our formulation of semistrict higher gauge theory, the fields are $\mathfrak{v}$–valued

and their global properties are controlled by a set of Čech gluing data rooted in a higher $\mathrm{Aut}(\mathfrak{v})$–valued cocycle acting by higher gauge transformations and obeying higher coherence conditions. The theory, then, is formulated in terms of $\mathfrak{v}$ only, conveniently seen as a 2–term L_∞ algebra, and its topological features are implicitly contained in the data. This is the line of thought followed below. As we shall see in due course, this way of proceeding works quite well for perturbative Lagrangian field theory, but it is of little us at the non perturbative level.

In this section, we shall first analyze the local aspects of 2–term L_∞ algebra gauge theory, neglecting global issues altogether. Later, we shall tackle the problem of assembling locally defined gauge theoretic data in a globally consistent manner by means of suitable gluing data. We shall also point out the strengths and weaknesses of our approach and endeavour to relate our formulation with others which have appeared in the literature.

3.1 2–term L_∞ algebra cohomology

The Chevalley–Eilenberg complex of a Lie algebra $\mathfrak{g}$ encodes the structure of $\mathfrak{g}$ [61]. It also abstracts the formal algebraic properties of the flat connections of a principal G–bundle, where G is a Lie group integrating $\mathfrak{g}$. The Weil complex of $\mathfrak{g}$ extends the Chevalley–Eilenberg complex in that it encapsulates the algebraic properties of the connections and curvatures thereof of a G–principal bundle and constitute the basic framework of the Chern–Weil theory of characteristic classes [62–64] (see also [65]). These well–known classical facts generalize to Lie 2–algebras and principal 2–bundle and beyond as worked out in refs. [35, 36]. We review these matters in this subsection and the next one, but we shall not go through the Chern–Weil theory which, albeit very important on its own, lies outside the scope of this paper.

Let $\mathfrak{g}$ be a Lie algebra. The *Chevalley–Eilenberg algebra* $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$ is the

graded commutative algebra $S(\mathfrak{g}^\vee[1]) \simeq \bigwedge^* \mathfrak{g}^\vee$ generated by $\mathfrak{g}^\vee[1]$, the 1 step degree shifted dual of $\mathfrak{g}$ [4]. The *Chevalley–Eilenberg differential* $\mathcal{Q}_{\mathrm{CE}(\mathfrak{g})}$ is the degree 1 differential defined as follows [5]. Let $\{e_a\}$ be a basis of $\mathfrak{g}$ and let $\{\pi^a\}$ be the basis of $\mathfrak{g}^\vee[1]$ dual to $\{e_a\}$. Set

$$\pi = \pi^a \otimes e_a, \tag{3.1.1}$$

Then, $\mathcal{Q}_{\mathrm{CE}(\mathfrak{g})}$ is given compactly by

$$\mathcal{Q}_{\mathrm{CE}(\mathfrak{g})}\pi = -\frac{1}{2}[\pi, \pi]. \tag{3.1.2}$$

It is immediately verified that $\mathcal{Q}_{\mathrm{CE}(\mathfrak{g})}$ is nilpotent,

$$\mathcal{Q}_{\mathrm{CE}(\mathfrak{g})}{}^2 = 0, \tag{3.1.3}$$

$(\mathrm{CE}(\mathfrak{g}), \mathcal{Q}_{\mathrm{CE}(\mathfrak{g})})$ is a so cochain complex. Its cohomology $H^*_{CE}(\mathfrak{g})$ is the *Chevalley–Eilenberg cohomology*, also known as *Lie algebra cohomology*, of $\mathfrak{g}$.

The nilpotence of $\mathcal{Q}_{\mathrm{CE}(\mathfrak{g})}$ is equivalent to the bracket $[\cdot, \cdot]$ satisfying the Jacobi identity, as is readily verified. Indeed, there is a one–to–one correspondence between Lie algebra structures on a vector space $\mathfrak{g}$ and nilpotent degree 1 differentials $\mathcal{Q}$ on $S(\mathfrak{g}^\vee[1])$.

As it is apparent from (3.1.1), (3.1.2), the Chevalley–Eilenberg complex $\mathrm{CE}(\mathfrak{g})$ formalizes the algebraic properties of the flat connections of a principle G–bundle, where $\mathrm{Lie}\, G = \mathfrak{g}$. (The precise meaning of this statement will be explained in the subsection.) The structure which does the same job for all connections of the G–bundle is the Weil complex of $\mathfrak{g}$, which we define next.

---

[4] For any vector space X, $X[q]$ is X itself with Grassmann degree shifted of q units. If q is odd, $S(X[q])$ is isomorphic to the exterior algebra $\bigwedge^* X$ of X with X in degree q. If q is even, $S(X[q])$ is isomorphic to the customary symmetric algebra $\bigvee^* X$ of X with X in degree q.

[5] A degree p differential d on a graded commutative algebra A is a vector endomorphism $d : A \to A$ such that $d(ab) = dab + (-1)^{p \deg b} adb$.

The *Weil algebra* $\mathrm{W}(\mathfrak{g})$ of $\mathfrak{g}$ is the graded commutative algebra $S(\mathfrak{g}^\vee[1] \oplus \mathfrak{g}^\vee[2]) \simeq \bigwedge^*(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee[1])$ generated by $\mathfrak{g}^\vee[1] \oplus \mathfrak{g}^\vee[2]$. The *Weil differential* $\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}$ is defined as follows. Let again $\{e_a\}$ be a basis of $\mathfrak{g}$ and let $\{\pi^a\}$, $\{\gamma^a\}$ be the bases of $\mathfrak{g}^\vee[1]$, $\mathfrak{g}^\vee[2]$ dual to $\{e_a\}$, respectively. We define π as in (3.1.1) and set

$$\gamma = \gamma^a \otimes e_a. \tag{3.1.4}$$

Then, $\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}$ is given by

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}\pi = -\frac{1}{2}[\pi, \pi] + \gamma, \tag{3.1.5a}$$

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}\gamma = -[\pi, \gamma]. \tag{3.1.5b}$$

It is readily checked that

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}{}^2 = 0. \tag{3.1.6}$$

$(\mathrm{W}(\mathfrak{g}), \mathcal{Q}_{\mathrm{W}(\mathfrak{g})})$ is a so cochain complex. Its cohomology $H^*_W(\mathfrak{g})$ turns out to be trivial in positive degree [6].

Requiring that $\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}\big|_{\mathfrak{g}^\vee[1]} = \mathcal{Q}_{\mathrm{CE}(\mathfrak{g})} + \sigma$, where $\sigma : \mathfrak{g}^\vee[1] \to \mathfrak{g}^\vee[2]$ is the degree shift vector morphism, and that $\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}$ is nilpotent fully determines the Weil differential $\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}$. Further, the projection $\mathfrak{g}^\vee[1] \oplus \mathfrak{g}^\vee[2] \to \mathfrak{g}^\vee[1]$ induces a canonical differential graded commutative algebra epimorphism $\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$.

From (3.1.1), (3.1.4), (3.1.5), it is apparent that the Weil complex $\mathrm{W}(\mathfrak{g})$ formalizes the algebraic properties of the connections of a principle G–bundle, (3.1.5a), (3.1.5b) corresponding to the definition of the curvature of a connection and to the Bianchi identity this obeys, respectively.

The above superalgebraic construction generalizes straightforwardly to Lie 2–algebras. Let $\mathfrak{v}$ be such an algebra seen as a 2–term L_∞ algebra. Similarly to ordi-

[6] Adding contractions I_x along the Lie algebra elements $x \in \mathfrak{g}$ to $\mathcal{Q}_{\mathrm{W}(\mathfrak{g})}$, it is possible to define the basic cohomology $H^*_{W\mathrm{bas}}(\mathfrak{g})$ of $\mathrm{W}(\mathfrak{g})$, which is isomorphic to $S(\mathfrak{g}^\vee[2])_{\mathrm{inv}}$ and so generally non trivial. This plays a pivotal role in Chern–Weil theory.

nary Lie algebras, the *Chevalley–Eilenberg algebra* $\mathrm{CE}(\mathfrak{v})$ of $\mathfrak{v}$ is the graded commutative algebra $S(\mathfrak{v}^\vee[1]) \simeq \bigwedge^* \mathfrak{v}^\vee$ generated by $\mathfrak{v}^\vee[1]$. The *Chevalley–Eilenberg differential* $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ is the degree 1 differential defined as follows. Let $\{e_a\}$, $\{E_\alpha\}$ be bases of $\hat{\mathfrak{v}}_0$, $\hat{\mathfrak{v}}_1$ [7] and let $\{\pi^a\}$, $\{\mathit{\Pi}^\alpha\}$ be the bases of ${\mathfrak{v}_0}^\vee[1]$, ${\mathfrak{v}_1}^\vee[1]$ dual to $\{e_a\}$, $\{E_\alpha\}$, respectively. Define now

$$\pi = \pi^a \otimes e_a, \tag{3.1.7a}$$

$$\mathit{\Pi} = \mathit{\Pi}^\alpha \otimes E_\alpha, \tag{3.1.7b}$$

in analogy to (3.1.1). Then, $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ is given succinctly by

$$\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}\pi = -\frac{1}{2}[\pi, \pi] + \partial \mathit{\Pi}, \tag{3.1.8a}$$

$$\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}\mathit{\Pi} = -[\pi, \mathit{\Pi}] + \frac{1}{6}[\pi, \pi, \pi]. \tag{3.1.8b}$$

The form of $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ is determined by the requirement that it is nilpotent,

$${\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}}^2 = 0, \tag{3.1.9}$$

$(\mathrm{CE}(\mathfrak{v}), \mathcal{Q}_{\mathrm{CE}(\mathfrak{v})})$ is a so cochain complex. The associated Chevalley–Eilenberg cohomology $H^*_{CE}(\mathfrak{v})$ is the Lie 2–algebra cohomology of $\mathfrak{v}$ generalizing ordinary Lie algebra cohomology.

The nilpotence of $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ is equivalent to the brackets $\partial, [\cdot, \cdot], [\cdot, \cdot, \cdot]$ satisfying the relations (2.2.1) characterizing a 2–term L_∞ algebra [8]. Indeed, again as for

---

[7] Here and below, for a graded vector space X, $\hat{X}$ is X with its grading set to 0.

[8] The condition ${\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}}^2 = 0$ translates into the following relations equivalent to (2.2.1)

$$[\pi, \partial \mathit{\Pi}] - \partial[\pi, \mathit{\Pi}] = 0, \tag{3.1.10a}$$

$$[\partial \mathit{\Pi}, \mathit{\Pi}] = 0, \tag{3.1.10b}$$

$$3[\pi, [\pi, \pi]] - \partial[\pi, \pi, \pi] = 0, \tag{3.1.10c}$$

$$2[\pi, [\pi, \mathit{\Pi}]] - [[\pi, \pi], \mathit{\Pi}] - [\pi, \pi, \partial \mathit{\Pi}] = 0, \tag{3.1.10d}$$

$$4[\pi, [\pi, \pi, \pi]] - 6[\pi, \pi, [\pi, \pi]] = 0. \tag{3.1.10e}$$

ordinary Lie algebras, there is a one–to–one correspondence between 2–term L_∞ algebra structures on a graded vector space $\mathfrak{v} = \mathfrak{v}_0 \oplus \mathfrak{v}_1$ and nilpotent degree 1 differentials $\mathcal{Q}$ on $S(\mathfrak{v}^\vee[1])$.

Generalizing the Lie algebraic case, we can assume that the Chevalley–Eilenberg complex $\mathrm{CE}(\mathfrak{v})$ formalizes the algebraic properties of the flat 2–connections of a principle V–2–bundle, where V is the appropriate kind of 2–group having $\mathfrak{v}$ as infinitesimal counterpart, (3.1.7), (3.1.8) defining the flatness conditions.

The *Weil algebra* $\mathrm{W}(\mathfrak{v})$ of $\mathfrak{v}$ is the graded commutative algebra $S(\mathfrak{v}^\vee[1] \oplus \mathfrak{v}^\vee[2]) \simeq \bigwedge^*(\mathfrak{v}^\vee \oplus \mathfrak{v}^\vee[1])$ generated by $\mathfrak{v}^\vee[1] \oplus \mathfrak{v}^\vee[2]$. The *Weil differential* $\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}$ is defined as follows. Let again $\{e_a\}$, $\{E_\alpha\}$ be bases of $\hat{\mathfrak{v}}_0$, $\hat{\mathfrak{v}}_1$ and let $\{\pi^a\}$, $\{\gamma^a\}$, $\{\mathit{\Pi}^\alpha\}$, $\{\mathit{\Gamma}^\alpha\}$ be the bases of ${\mathfrak{v}_0}^\vee[1]$, ${\mathfrak{v}_0}^\vee[2]$, ${\mathfrak{v}_1}^\vee[1]$, ${\mathfrak{v}_1}^\vee[2]$ dual to $\{e_a\}$, $\{E_\alpha\}$, respectively. We define π, $\mathit{\Pi}$ as in (3.1.7) and set

$$\gamma = \gamma^a \otimes e_a, \tag{3.1.11a}$$

$$\mathit{\Gamma} = \mathit{\Gamma}^\alpha \otimes E_\alpha, \tag{3.1.11b}$$

Then, $\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}$ is given by

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}\pi = -\frac{1}{2}[\pi,\pi] + \partial\mathit{\Pi} + \gamma, \tag{3.1.12a}$$

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}\mathit{\Pi} = -[\pi,\mathit{\Pi}] + \frac{1}{6}[\pi,\pi,\pi] + \mathit{\Gamma}, \tag{3.1.12b}$$

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}\gamma = -[\pi,\gamma] - \partial\mathit{\Gamma}, \tag{3.1.12c}$$

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}\mathit{\Gamma} = -[\pi,\mathit{\Gamma}] + [\gamma,\mathit{\Pi}] - \frac{1}{2}[\pi,\pi,\gamma]. \tag{3.1.12d}$$

Again, it is checked that

$$\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}{}^2 = 0. \tag{3.1.13}$$

$(\mathrm{W}(\mathfrak{v}), \mathcal{Q}_{\mathrm{W}(\mathfrak{v})})$ is a so cochain complex. Its cohomology $H^*_W(\mathfrak{v})$ turns out to be trivial in positive degree as for ordinary Lie algebras.

Again, requiring that $\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}\big|_{\mathfrak{v}^\vee[1]} = \mathcal{Q}_{\mathrm{CE}(\mathfrak{v})} + \sigma$, where $\sigma : \mathfrak{v}^\vee[1] \to \mathfrak{v}^\vee[2]$ is the

degree shift vector morphism, and that $\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}$ is nilpotent fully determines the Weil differential $\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}$. Further, the projection $\mathfrak{v}^\vee[1] \oplus \mathfrak{v}^\vee[2] \to \mathfrak{v}^\vee[1]$ induces a canonical differential graded commutative algebra epimorphism $\mathrm{W}(\mathfrak{v}) \to \mathrm{CE}(\mathfrak{v})$.

Extending the Lie algebraic framework once more, we can think of the Weil complex $\mathrm{W}(\mathfrak{v})$ as an algebraic model describing the 2–connections of a principle V–2–bundle, (3.1.12a), (3.1.12b) corresponding to the definition of the curvature components and (3.1.12c), (3.1.12d) expressing the Bianchi identities which these obey.

3.2 2–term L_∞ algebra gauge theory, local aspects

In ordinary as well as higher gauge theory, fields propagate on a fixed d–fold M. Each field is characterized by its *form degree* m and *ghost number degree* n for some integers $m \geq 0$ and n. In that case, the field is said to have *bidegree* (m, n).

To study the local aspects of the theory, we first assume that M is diffeomorphic to $\mathbb{R}^d$. On such an M, a field of bidegree (m, n) is then an element of the space $\Omega^m(M, E[n])$ of m–forms on M with values in $E[n]$, where E is some vector space. In an ordinary gauge theory with structure Lie algebra $\mathfrak{g}$, fields are generally drawn from the spaces $\Omega^m(M, \mathfrak{g}[n])$ and $\Omega^m(M, \mathfrak{g}^\vee[n])$. In the first case, they are called bidegree (m, n) fields, in the second, bidegree (m, n) dual fields.

The main field of the gauge theory is the *connection* ω, which is a bidegree $(1, 0)$ field. ω is characterized by its *curvature* f, the bidegree $(2, 0)$ field given by

$$f = d\omega + \frac{1}{2}[\omega, \omega]. \tag{3.2.1}$$

f satisfies the standard *Bianchi identity*

$$df + [\omega, f] = 0. \tag{3.2.2}$$

It is a classic result that the assignment of a connection ω is equivalent to that

of a differential graded commutative algebra morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(M)$ from the Weil algebra of $\mathfrak{g}$ (cf. subsect. 3.1) to the differential forms of M. The morphism is the one mapping the generators π, γ of $\mathrm{W}(\mathfrak{g})$ respectively in ω, f and its being differential is evident from comparing eqs. (3.1.5) with eqs. (3.2.1), (3.2.2).

The connection ω is *flat* if the curvature $f = 0$. This happens precisely when the associated morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(M)$ factors as $\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g}) \to \Omega^*(M)$, where $\mathrm{W}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$ is the canonical morphism of the Weil onto the Chevalley–Eilenberg algebra of $\mathfrak{g}$ (cf. subsect. 3.1), as follows from (3.1.2) and (3.2.1).

The *covariant derivative* of a field ϕ is given by the well–known expression

$$D\phi = d\phi + [\omega, \phi] \tag{3.2.3}$$

and satisfies the standard Ricci identity

$$DD\phi = [f, \phi]. \tag{3.2.4}$$

The covariant derivative of a dual field υ is given similarly by

$$D\upsilon = d\upsilon + [\omega, \upsilon]^\vee, \tag{3.2.5}$$

the Ricci identity being

$$DD\upsilon = [f, \upsilon]^\vee \tag{3.2.6}$$

[9]. The covariant derivative preserves the canonical pairing of fields and dual fields

$$d\langle \upsilon, \phi\rangle = \langle D\upsilon, \phi\rangle + (-1)^{r+s}\langle \upsilon, D\phi\rangle, \tag{3.2.9}$$

[9] Using the canonical duality pairing $\langle \cdot, \cdot\rangle$ of $\mathfrak{g}^\vee, \mathfrak{g}$, we define the dual brackets in $\mathfrak{g}^\vee$ by

$$\langle [x, \xi]^\vee, z\rangle = -\langle \xi, [x, z]\rangle. \tag{3.2.7}$$

Similarly, we can associate with any automorphism ϕ of $\mathfrak{g}$ its dual automorphism $\phi^\vee$ of $\mathfrak{g}^\vee$ by

$$\langle \phi^\vee(\xi), x\rangle = \langle \xi, \phi^{-1}(x)\rangle, \tag{3.2.8}$$

if υ has bidegree (r, s).

The Bianchi identity (3.2.2) obeyed f can be written compactly as

$$Df = 0. \tag{3.2.10}$$

The Bianchi identity, so, contains information sufficient to recover the form of the covariant differentiation operator D on fields. Imposing that (3.2.9) holds determines the form of D on dual fields.

The familiar properties of connections recalled above provide us with important clues about the definition of 2–connection appropriate for semistrict higher gauge theory and suggest how to construct the covariant derivative operator in such context. This, we shall do next. Our treatment is actually a particular case of the general formulation of [35, 36]. (See subsect. 3.9 for a further discussion.)

In 2–term L_∞ algebra gauge theory, as a rule, fields organize in *field doublets* $(\phi, \varPhi) \in \Omega^m(M, \hat{\mathfrak{v}}_0[n]) \times \Omega^{m+1}(M, \hat{\mathfrak{v}}_1[n])$ and *dual field doublets* $(\varUpsilon, \upsilon) \in \Omega^m(M, \hat{\mathfrak{v}}_1{}^\vee[n]) \times \Omega^{m+1}(M, \hat{\mathfrak{v}}_0{}^\vee[n])$, where $-1 \leq m \leq d$ (see fn. 7 for the definition of the hat notation). If $m = -1$, the first component of the doublet vanishes. If $m = d$, the second component does. The doublets of this form are said to have bidegree (m, n).

There is a distinguished field doublet in the theory, the *connection doublet* $(\omega, \varOmega)$ of bidegree $(1, 0)$. Associated with it is the *curvature doublet* (f, F) of bidegree $(2, 0)$ defined by the expressions

$$f = d\omega + \frac{1}{2}[\omega, \omega] - \partial\varOmega, \tag{3.2.11a}$$

$$F = d\varOmega + [\omega, \varOmega] - \frac{1}{6}[\omega, \omega, \omega]. \tag{3.2.11b}$$

From (3.2.11), it is readily verified that (f, F) satisfies the *Bianchi identities*

$$df + [\omega, f] + \partial F = 0, \tag{3.2.12a}$$

$$dF + [\omega, F] - [f, \varOmega] + \frac{1}{2}[\omega, \omega, f] = 0 \tag{3.2.12b}$$

analogous to the Bianchi identity (3.2.2) of ordinary gauge theory.

The above definition is justified by the request that the assignment of a connection doublet be equivalent to that of a differential graded commutative algebra morphism $\mathrm{W}(\mathfrak{v}) \to \Omega^*(M)$ from the Weil algebra of $\mathfrak{v}$ (cf. subsect. 3.1) to the differential forms of M, generalizing the corresponding property of connections in ordinary gauge theory. The morphism is the one mapping the generators π, Π γ, Γ of $\mathrm{W}(\mathfrak{v})$ respectively to ω, Ω, f, F and its differential property is evident from the comparison of eqs. (3.1.12) with eqs. (3.2.11), (3.2.12).

The connection doublet (ω, Ω) is said *flat* if the curvature doublet $(f, F) = (0, 0)$, with an obvious naming. (ω, Ω) is flat precisely when the associated morphism $\mathrm{W}(\mathfrak{v}) \to \Omega^*(M)$ factors as $\mathrm{W}(\mathfrak{v}) \to \mathrm{CE}(\mathfrak{v}) \to \Omega^*(M)$, where $\mathrm{W}(\mathfrak{v}) \to \mathrm{CE}(\mathfrak{v})$ is the canonical morphism of the Weil onto the Chevalley–Eilenberg algebra of $\mathfrak{v}$ (cf. subsect. 3.1), generalizing again the corresponding property of ordinary connections, as it is apparent from inspecting eqs. (3.1.8) and (3.2.11).

Let (ϕ, Φ) be a field doublet of bidegree (p, q). The *covariant derivative doublet* of (ϕ, Φ) is the field doublet $(D\phi, D\Phi)$ of bidegree $(p + 1, q)$ defined by

$$D\phi = d\phi + [\omega, \phi] + (-1)^{p+q}\partial\Phi, \tag{3.2.13a}$$

$$D\Phi = d\Phi + [\omega, \Phi] - (-1)^{p+q}[\phi, \Omega] + \frac{(-1)^{p+q}}{2}[\omega, \omega, \phi]. \tag{3.2.13b}$$

The sign $(-1)^{p+q}$ is conventional, since the relative sign of ϕ, Φ cannot be fixed in any natural manner. From (3.2.13), we deduce easily the appropriate version of the Ricci identities,

$$DD\phi = [f, \phi], \tag{3.2.14a}$$

$$DD\Phi = [f, \Phi] - [\phi, F] - [\phi, \omega, f]. \tag{3.2.14b}$$

The explicit apparence of the connection component ω in the right hand side of (3.2.14b) is a consequence of the presence of a term quadratic in ω in (3.2.13b).

Let $(\varUpsilon, \upsilon)$ be a dual field doublet of bidegree (r,s). The *covariant derivative dual doublet* of $(\varUpsilon, \upsilon)$ is the dual field doublet $(D\varUpsilon, D\upsilon)$ of bidegree $(r+1,s)$ defined by

$$D\varUpsilon = d\varUpsilon + [\omega, \varUpsilon]^\vee - (-1)^{r+s}\partial^\vee\upsilon, \tag{3.2.15a}$$

$$D\upsilon = d\upsilon + [\omega, \upsilon]^\vee - (-1)^{r+s}[\varOmega, \varUpsilon]^\vee - \frac{(-1)^{r+s}}{2}[\omega, \omega, \varUpsilon]^\vee, \tag{3.2.15b}$$

analogously to (3.2.13) [10]. The Ricci identities then read as

$$DD\varUpsilon = [f, \varUpsilon]^\vee, \tag{3.2.18a}$$

$$DD\upsilon = [f, \upsilon]^\vee - (-1)^{r+s}[F, \varUpsilon]^\vee - (-1)^{r+s}[f, \omega, \varUpsilon]^\vee, \tag{3.2.18b}$$

---

[10] Using the canonical duality pairing $\langle\cdot,\cdot\rangle$ of $\hat{\mathfrak{v}}_0{}^\vee, \hat{\mathfrak{v}}_0$ and $\hat{\mathfrak{v}}_1{}^\vee, \hat{\mathfrak{v}}_1$, we obtain a canonical 2*–term L_∞ algebra costructure*. This consists of the linear maps $\partial^\vee : \hat{\mathfrak{v}}_0{}^\vee \to \hat{\mathfrak{v}}_1{}^\vee$, $[\cdot,\cdot]^\vee : \hat{\mathfrak{v}}_0 \otimes \hat{\mathfrak{v}}_0{}^\vee \to \hat{\mathfrak{v}}_0{}^\vee$, $[\cdot,\cdot]^\vee : \hat{\mathfrak{v}}_0 \otimes \hat{\mathfrak{v}}_1{}^\vee \to \hat{\mathfrak{v}}_1{}^\vee$, $[\cdot,\cdot]^\vee : \hat{\mathfrak{v}}_1 \otimes \hat{\mathfrak{v}}_1{}^\vee \to \hat{\mathfrak{v}}_0{}^\vee$, $[\cdot,\cdot,\cdot]^\vee : (\hat{\mathfrak{v}}_0 \wedge \hat{\mathfrak{v}}_0) \otimes \hat{\mathfrak{v}}_1{}^\vee \to \hat{\mathfrak{v}}_0{}^\vee$ defined by

$$\langle \partial^\vee \xi, X\rangle = \langle \xi, \partial X\rangle, \tag{3.2.16a}$$

$$\langle [x,\xi]^\vee, z\rangle = -\langle \xi, [x,z]\rangle, \tag{3.2.16b}$$

$$\langle [x,\varXi]^\vee, Z\rangle = -\langle \varXi, [x,Z]\rangle, \tag{3.2.16c}$$

$$\langle [X,\varXi]^\vee, z\rangle = +\langle \varXi, [z,X]\rangle, \tag{3.2.16d}$$

$$\langle [x,y,\varXi]^\vee, z\rangle = -\langle \varXi, [x,y,z]\rangle, \tag{3.2.16e}$$

where $\xi \in \hat{\mathfrak{v}}_0{}^\vee$, $\varXi \in \hat{\mathfrak{v}}_1{}^\vee$ Again, we use the notation $[\cdot,\cdot]^\vee$ for all 2–argument cobrackets.

Using the duality pairing again, we can associate with any automorphism $\phi = (\phi_0, \phi_1, \phi_2)$ of $\mathfrak{v}$ its *dual automorphism* of ϕ. This consists of the linear maps ${\phi^\vee}_0 : \hat{\mathfrak{v}}_0{}^\vee \to \hat{\mathfrak{v}}_0{}^\vee$, ${\phi^\vee}_1 : \hat{\mathfrak{v}}_1{}^\vee \to \hat{\mathfrak{v}}_1{}^\vee$, ${\phi^\vee}_2 : \hat{\mathfrak{v}}_0 \otimes \hat{\mathfrak{v}}_1{}^\vee \to \hat{\mathfrak{v}}_0{}^\vee$ defined by the relations

$$\langle {\phi^\vee}_0(\xi), x\rangle = \langle \xi, {\phi^{-1}}_0(x)\rangle, \tag{3.2.17a}$$

$$\langle {\phi^\vee}_1(\Xi), X\rangle = \langle \Xi, {\phi^{-1}}_1(X)\rangle, \tag{3.2.17b}$$

$$\langle {\phi^\vee}_2(x,\Xi), y\rangle = \langle \Xi, {\phi^{-1}}_2(x,y)\rangle. \tag{3.2.17c}$$

We shall denote the dual of ϕ by $\phi^\vee$ or $({\phi^\vee}_0, {\phi^\vee}_1, {\phi^\vee}_2)$.

analogously to (3.2.14).

There exists a natural pairing of field and dual field doublets. The pairing of a field doublet (ϕ, Φ) of bidegree (p, q) and a dual field doublet (Υ, υ) of bidegree (r, s) is a the scalar valued field of bidegree $(p + r + 1, q + s)$ given by

$$\langle(\Upsilon, \upsilon), (\phi, \Phi)\rangle = \langle\upsilon, \phi\rangle - (-1)^{p+q}\langle\Upsilon, \Phi\rangle. \tag{3.2.19}$$

The basic property of the pairing is that

$$d\langle(\Upsilon, \upsilon), (\phi, \Phi)\rangle = \langle(D\Upsilon, D\upsilon), (\phi, \Phi)\rangle - (-1)^{r+s}\langle(\Upsilon, \upsilon), (D\phi, D\Phi)\rangle. \tag{3.2.20}$$

By the Stokes' theorem, upon integration on M, this relation yields an integration by parts formula for the covariant derivative of (dual) field doublets.

The above definition of covariant differentiation is yielded by the request that the Bianchi identities (3.2.12) be expressed as the vanishing of the covariant derivative doublet (Df, DF) of the curvature doublet (f, F)

$$Df = 0, \tag{3.2.21a}$$

$$DF = 0 \tag{3.2.21b}$$

as it is the case for the Bianchi identity of ordinary gauge theory, eq. (3.2.10). Imposing that (3.2.20) holds determines the the action of D on dual fields.

3.3 The 2–group of 2–term L_∞ algebra gauge transformations

One expects that there is a notion of semistrict higher gauge transformation generalizing the corresponding notion of ordinary gauge theory and that this plays an important role in semistrict higher gauge theory. This is indeed so.

In fact, 2–term L_∞ algebra gauge transformations can be meaningfully defined. Further, they organize in an infinite dimensional strict 2–group described below (cf. subsect. 2.7). We assume again that M is diffeomorphic to $\mathbb{R}^d$.

The definition of 2–term L_∞ algebra 1–gauge transformation given below is not straightforward and needs to be justified. To this end, we begin by considering an ordinary gauge theory with structure Lie algebra $\mathfrak{g}$. A gauge transformation is a map $g \in \mathrm{Map}(M, \mathrm{Aut}(\mathfrak{g}))$ of a special form: its range consists of inner automorphisms of $\mathfrak{g}$. So, letting G be a Lie group integrating $\mathfrak{g}$, there is a map $\gamma \in \mathrm{Map}(M, G)$ such that $g(x) = \mathrm{Ad}\,\gamma(x) = \gamma x \gamma^{-1}$. When we try to define a gauge transformation in a 2–term L_∞ algebra gauge theory with structure algebra $\mathfrak{v}$ following the same line, we soon run into trouble, as there is no natural notion of inner automorphism of $\mathfrak{v}$.

We circumvent this difficulty as follows. We note that $\sigma_g = \gamma^{-1} d\gamma$ is a flat connection such that $dg(x) = g([\sigma_g, x])$. Thus, we may extend the notion of gauge transformation by defining it as a pair of

1. a map $g \in \mathrm{Map}(M, \mathrm{Aut}(\mathfrak{g}))$,
2. a flat connection σ_g,
$$d\sigma_g + \frac{1}{2}[\sigma_g, \sigma_g] = 0, \tag{3.3.1}$$
3. related to g through the condition
$$g^{-1} dg(x) - [\sigma_g, x] = 0. \tag{3.3.2}$$

We shall denote the gauge transformation by (g, σ_g) or simply by g, having in mind that now σ_g is not determined by g but participates with g in the transformation. Further, we shall denote by $\mathrm{Gau}(M, \mathfrak{g})$ the set of all such extended gauge transformations.

Albeit not all $g \in \mathrm{Gau}(M, \mathfrak{g})$ correspond to conventional gauge transformations [11], the above observation provides clues which indicate the direction along

[11] If $g \in \mathrm{Map}(M, \mathrm{Aut}(\mathfrak{g}))$ has the property that there is a flat connection σ_g such that

which to construct the generalization of the notion of gauge transformation appropriate for 2–term L_∞ algebra gauge theory.

A 2*–term* L_∞ *algebra* 1*–gauge transformation* consists of the following set of data.

1. a map $g \in \mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$ (cf. sect. 2.9);

2. a flat connection doublet $(\sigma_g, \varSigma_g)$,

$$d\sigma_g + \frac{1}{2}[\sigma_g, \sigma_g] - \partial \varSigma_g = 0, \tag{3.3.3a}$$

$$d\varSigma_g + [\sigma_g, \varSigma_g] - \frac{1}{6}[\sigma_g, \sigma_g, \sigma_g] = 0; \tag{3.3.3b}$$

3. an element τ_g of $\Omega^1(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$ satisfying

$$\begin{aligned} d\tau_g(x) + [\sigma_g, \tau_g(x)] - [x, \varSigma_g] + \frac{1}{2}[\sigma_g, \sigma_g, x] \\ + \tau_g([\sigma_g, x] + \partial \tau_g(x)) = 0. \end{aligned} \tag{3.3.4}$$

These data are required to satisfy the following relations. If $g = (g_0, g_1, g_2)$ (cf. sect. 2.3), then one has

$$g_0{}^{-1} dg_0(x) - [\sigma_g, x] - \partial \tau_g(x) = 0, \tag{3.3.5a}$$

$$g_1{}^{-1} dg_1(X) - [\sigma_g, X] - \tau_g(\partial X) = 0, \tag{3.3.5b}$$

$$\begin{aligned} g_1{}^{-1}(dg_2(x, y) - g_2(g_0{}^{-1} dg_0(x), y) - g_2(x, g_0{}^{-1} dg_0(y))) \\ - [\sigma_g, x, y] - \tau_g([x, y]) + [x, \tau_g(y)] - [y, \tau_g(x)] = 0 \end{aligned} \tag{3.3.5c}$$

hold. In the following, we are going to denote a 2–term L_∞ algebra 1–gauge transformation such as the above as $(g, \sigma_g, \varSigma_g, \tau_g)$ or simply as g. We remark

---

$g^{-1} dg(x) = [\sigma_g, x]$ for $x \in \mathfrak{g}$, then there is a map $\gamma \in \mathrm{Map}(M, G)$ and a constant $g_0 \in \mathrm{Aut}(\mathfrak{g})$ such that $g = g_0 \,\mathrm{Ad}\,\gamma$. In general, $g_0 \neq 1_\mathfrak{g}$. Thus, the range of g does not necessarily consist of inner automorphisms of $\mathfrak{g}$.

that, in so doing, we are not implying that σ_g, Σ_g, τ_g are determined by g, but only that they are the partners of g in the gauge transformation. We shall denote the set of all 2–term L_∞ algebra 1–gauge transformations by $\mathrm{Gau}_1(M, \mathfrak{v})$.

The remarks made at the beginning of this subsection already justify to a considerable extent the definition of 2–term L_∞ algebra 1–gauge transformation given above. When the Lie algebra $\mathfrak{g}$ gets replaced by a more general 2–term L_∞ algebra $\mathfrak{v}$, the flat connection σ_g gets promoted to a flat connection doublet (σ_g, Σ_g) as is appropriate. We obtain in this way eqs. (3.3.3). The point is that this is not sufficient to fully explain the form of relations (3.3.5), for reasons explained next.

For the ordinary gauge transformation considered above, in order the Maurer–Cartan equation $d(g^{-1}dg) + g^{-1}dgg^{-1}dg = 0$ to be satisfied, it is sufficient that σ_g is flat. The proof of this requires crucially the use of the Jacobi identity of the Lie algebra $\mathfrak{g}$. When $\mathfrak{g}$ is replaced by a 2–term L_∞ algebra $\mathfrak{v}$, that identity is no longer available. This forces one to introduce another object, namely τ_g, and modify the naive relations $g_0^{-1}dg_0(x) = [\sigma_g, x]$, $g_1^{-1}dg_1(X) = [\sigma_g, X]$, as shown in (3.3.5a), (3.3.5b). If τ_g vanished, for the the Maurer–Cartan equations $d({g_0}^{-1}dg_0) + {g_0}^{-1}dg_0{g_0}^{-1}dg_0 = 0$, $d({g_1}^{-1}dg_1) + {g_1}^{-1}dg_1{g_1}^{-1}dg_1 = 0$ to be satisfied, the flatness relations (3.3.3) would not be sufficiently by themselves: one would need an extra condition, namely $-[x, \Sigma_g] + \frac{1}{2}[\sigma_g, \sigma_g, x] = 0$. This latter, a purely algebraic requirement on the flat connection doublet (σ_g, Σ_g), does not fit into our higher gauge theoretic set-up in any natural way, and, so, it is hardly acceptable. Once we allow for τ_g, however, this condition takes the natural form of a differential consistency relation satisfied by τ_g, viz (3.3.4). The reasoning just expounded justifies calling (3.3.4) "2–Maurer–Cartan equation". As to relation (3.3.5c), it is just a natural coherence condition ensuring the compatibility of (3.3.5a), (3.3.5b) and (2.3.1).

For any two 1–gauge transformations $g, h \in \mathrm{Gau}_1(M, \mathfrak{v})$, a 2*–term* L_∞ *algebra*

2–*gauge transformation from* g *to* h consists of the following data.

1. a map $F \in \mathrm{Map}(M, \mathrm{Aut}_2(\mathfrak{v}))(g, h)$, where $\mathrm{Map}(M, \mathrm{Aut}_2(\mathfrak{v}))(g, h)$ is the space of sections of the fiber bundle $\bigcup_{m \in M} \mathrm{Aut}_2(\mathfrak{v})(g(m), h(m)) \to M$ (cf. sect. 2.9);

2. an element $A_F \in \Omega^1(M, \hat{\mathfrak{v}}_1)$.

They are required to satisfy the following relations,

$$\sigma_g - \sigma_h = \partial A_F, \tag{3.3.6a}$$

$$\Sigma_g - \Sigma_h = dA_F + [\sigma_h, A_F] + \frac{1}{2}[\partial A_F, A_F], \tag{3.3.6b}$$

$$\tau_g(x) - \tau_h(x) = [x, A_F] + {g_1}^{-1}\big(dF(x) - F([\sigma_h, x] + \partial\tau_h(x))\big). \tag{3.3.6c}$$

In the following, we are going to denote a 2–term L_∞ algebra 2–gauge transformation such as the above as (F, A_F) or simply as F. Again, in so doing, we are not implying that A_F is determined by F, but only that it is the partner of F in the gauge transformation. We shall also write $F : g \Rightarrow h$ to emphasize its source and target. We shall denote the set of all 2–term L_∞ algebra 2–gauge transformations $F : g \Rightarrow h$ by $\mathrm{Gau}_2(M, \mathfrak{v})(g, h)$ and the set of all 2–gauge transformations F by $\mathrm{Gau}_2(M, \mathfrak{v})$.

To justify the above definition of 2–term L_∞ algebra 2–gauge transformation, the following remarks are in order. Suppose that $(g, \sigma_g, \Sigma_g, \tau_g)$ is a 2–term L_∞ algebra 1–gauge transformation. Let us ask what the most natural class of deformations of $(g, \sigma_g, \Sigma_g, \tau_g)$ which preserve its being a 1–gauge transformation and which can be parametrized in terms of elementary fields is. As $g, h \in \mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$, it is natural to demand that g, h are the source and the target of some $F \in \mathrm{Map}(M, \mathrm{Aut}_2(\mathfrak{v}))(g, h)$. Once this is done, the only remaining deformational degree of freedom is an element $A \in \Omega^1(M, \hat{\mathfrak{v}}_1)$ turning σ_g into $\sigma_h = \sigma_g - \partial A$. We require A to be $\hat{\mathfrak{v}}_1$–valued in order it to be utilizable to deform

Σ_g into $\Sigma_h = \Sigma_g - dA + \frac{1}{2}[\partial A, A] + \cdots$ and $\tau_g(x)$ into $\tau_h(x) = \tau_g(x) - [x, A] + \cdots$. Requiring that $(h, \sigma_h, \Sigma_h, \tau_h)$ is a 1–gauge transformation fixes the form of the terms not shown.

In ordinary gauge theory, gauge transformations form a group, the guage group of the gauge theory. This remains true also for the more general gauge transformations, which we have defined at the beginning of this subsection (cf. eqs. (3.3.1), (3.3.2)). Define

$$h \diamond g = hg, \tag{3.3.7a}$$

$$\sigma_{h \diamond g} = \sigma_g + g^{-1}(\sigma_h), \tag{3.3.7b}$$

$$g^{-1_\diamond} = g^{-1}, \tag{3.3.7c}$$

$$\sigma_{g^{-1_\diamond}} = -g(\sigma_g), \tag{3.3.7d}$$

$$i = \mathrm{id}_{\mathfrak{g}}, \tag{3.3.7e}$$

$$\sigma_i = 0, \tag{3.3.7f}$$

where $g, h \in \mathrm{Gau}(M, \mathfrak{g})$ and, in (3.3.7a), (3.3.7c), (3.3.7e), the composition, inversion and unit in the right hand side are those of $\mathrm{Aut}(\mathfrak{g})$ thought of as holding pointwise on M. Then, as it is immediately checked, $\mathrm{Gau}(M, \mathfrak{g})$ is an ordinary group, the extended gauge group of the theory. Inspection of (3.3.7) shows that $\mathrm{Gau}(M, \mathfrak{g})$ is a subgroup of the semidirect product $\Omega^1(M, \mathfrak{g}) \rtimes \mathrm{Map}(M, \mathrm{Aut}(\mathfrak{g}))$ associated with the right action of $\mathrm{Map}(M, \mathrm{Aut}(\mathfrak{g}))$ on $\Omega^1(M, \mathfrak{g})$ induced by the right action of $\mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{g}$. $\mathrm{Gau}(M, \mathfrak{g})$ is a proper subgroup, when M is not a point, because its elements satisfy the additional differential relations (3.3.1), (3.3.2).

The property which we have found to hold in ordinary gauge theory generalizes to 2–term L_∞ algebra gauge theory. Indeed, as we show below in detail, it is possible to define a composition and an inversion law and a unit in $\mathrm{Gau}_1(M, \mathfrak{v})$ and horizontal and vertical composition and inversion laws and units in $\mathrm{Gau}_2(M, \mathfrak{v})$,

making $\mathrm{Gau}(M,\mathfrak{v}) = (\mathrm{Gau}_1(M,\mathfrak{v}), \mathrm{Gau}_2(M,\mathfrak{v}))$ a strict 2–group (cf. subsect. 2.7).

The composition and inversion laws and the unit of 1–gauge transformation are defined by the relations

$$h \diamond g = h \circ g, \tag{3.3.8a}$$

$$\sigma_{h\diamond g} = \sigma_g + g_0{}^{-1}(\sigma_h), \tag{3.3.8b}$$

$$\Sigma_{h\diamond g} = \Sigma_g + g_1{}^{-1}\Big(\Sigma_h + \frac{1}{2}g_2(g_0{}^{-1}(\sigma_h), g_0{}^{-1}(\sigma_h))\Big) - \tau_g(g_0{}^{-1}(\sigma_h)), \tag{3.3.8c}$$

$$\tau_{h\diamond g}(x) = \tau_g(x) + g_1{}^{-1}\big(\tau_h(g_0(x)) - g_2(g_0{}^{-1}(\sigma_h), x)\big), \tag{3.3.8d}$$

$$g^{-1_\diamond} = g^{-1_\circ}, \tag{3.3.8e}$$

$$\sigma_{g^{-1_\diamond}} = -g_0(\sigma_g), \tag{3.3.8f}$$

$$\Sigma_{g^{-1_\diamond}} = -g_1(\Sigma_g + \tau_g(\sigma_g)) - \frac{1}{2}g_2(\sigma_g, \sigma_g), \tag{3.3.8g}$$

$$\tau_{g^{-1_\diamond}}(x) = -g_1(\tau_g(g_0{}^{-1}(x))) - g_2(\sigma_g, g_0{}^{-1}(x)), \tag{3.3.8h}$$

$$i = \mathrm{id}_{\mathfrak{v}}, \tag{3.3.8i}$$

$$\sigma_i = 0, \tag{3.3.8j}$$

$$\Sigma_i = 0, \tag{3.3.8k}$$

$$\tau_i(x) = 0, \tag{3.3.8l}$$

where $g, h \in \mathrm{Gau}_1(M,\mathfrak{v})$. In (3.3.8a), (3.3.8e), (3.3.8i), the composition, inversion and unit in the right hand side are those of $\mathrm{Aut}_1(\mathfrak{v})$ thought of as holding pointwise on M (cf. eqs. (2.3.3), (2.3.4), (2.9.1)).

The horizontal and vertical composition and inversion laws and the units of 2–gauge transformations are defined by the relations

$$G \diamond F(x) = G \circ F(x), \tag{3.3.9a}$$

$$A_{G\diamond F} = A_F + h^{-1}{}_1(A_G) - g_1{}^{-1}Fh_0{}^{-1}(\sigma_k), \tag{3.3.9b}$$

$$F^{-1_\diamond}(x) = F^{-1_\circ}(x), \tag{3.3.9c}$$

$$A_{F^{-1_\diamond}} = -g_1(A_F) - F(\sigma_h), \tag{3.3.9d}$$

$$K \bullet H(x) = K \cdot H(x), \tag{3.3.9e}$$

$$A_{K \bullet H} = A_H + A_K, \tag{3.3.9f}$$

$$H^{-1_\bullet}(x) = H^{-1_\cdot}(x), \tag{3.3.9g}$$

$$A_{H^{-1_\bullet}} = -A_H, \tag{3.3.9h}$$

$$I_g(x) = \mathrm{Id}_g(x), \tag{3.3.9i}$$

$$A_{I_g} = 0, \tag{3.3.9j}$$

where $g, h, k, l \in \mathrm{Gau}_1(M, \mathfrak{v})$ and $F, G, H, K \in \mathrm{Gau}_2(M, \mathfrak{v})$, with $F : g \Rightarrow h$, $G : k \Rightarrow l$ and H, K composible. In (3.3.9a), (3.3.9c), (3.3.9e), (3.3.9g), (3.3.9i), the horizontal and vertical composition and inversion and the units in the right hand side are those of $\mathrm{Aut}_2(\mathfrak{v})$ thought of as holding pointwise on M (cf. eqs. (2.3.5)–(2.3.7), (2.9.2), (2.9.3)). The expressions can be written in several other equivalent ways using repeatedly relations (2.3.2) with ϕ, ψ, Φ replaced by g, h, F or k, l, G.

The composition, inversion and unit structures just defined satisfy the axioms (2.7.1) rendering $(\mathrm{Gau}_1(M, \mathfrak{v}), \mathrm{Gau}_2(M, \mathfrak{v}))$ a strict 2–group, as announced. From (3.3.8), it appears that the 1–cell group $\mathrm{Gau}_1(M, \mathfrak{v})$ is a subgroup of the semidirect product $\Omega^1(M, \hat{\mathfrak{v}}_0) \oplus \Omega^2(M, \hat{\mathfrak{v}}_1) \oplus \Omega^1(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1)) \rtimes \mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$ associated with a certain right action of $\mathrm{Map}(M, \mathrm{Aut}_1(\mathfrak{v}))$ on $\Omega^1(M, \hat{\mathfrak{v}}_0) \oplus \Omega^2(M, \hat{\mathfrak{v}}_1) \oplus \Omega^1(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$. $\mathrm{Gau}_1(M, \mathfrak{v})$ is a proper subgroup, when M is not a point, as its elements satisfy the additional differential relations (3.3.3)–(3.3.5). These findings suggest that the full 2–group $\mathrm{Gau}(M, \mathfrak{v})$ may be a 2–subgroup of a conjectural semidirect product 2–group defined by the relations (3.3.8), (3.3.9). We shall not elaborate on this point any further.

3.4 The gauge transformation action

An important test of the viability of the definition of semistrict higher gauge transformation we have worked out in subsect. 3.3 is the existence of a suitable gauge transformation action on fields. In principle, several prescriptions are possible and a full inspection of all options is out of question. To select the appropriate definition, we proceed once more from standard gauge theory. Here, we assume again that M is diffeomorphic to $\mathbb{R}^d$.

Consider an ordinary gauge theory with structure Lie algebra $\mathfrak{g}$. Conventionally, a gauge transformation is a mapping $\gamma \in \mathrm{Map}(M, G)$, where G is a Lie group integrating $\mathfrak{g}$, acting on a connection ω by ${}^\gamma\omega = \gamma\omega\gamma^{-1} - d\gamma\gamma^{-1}$ and on its curvature f by ${}^\gamma f = \gamma f\gamma^{-1}$. As argued in subsect. 3.3, when aiming to construct higher generalizations, it is useful to extend the range of gauge transformations to all $g \in \mathrm{Gau}(M, \mathfrak{g})$. Noticing that, in the usual case just considered, $(g, \sigma_g) = (\mathrm{Ad}\,\gamma, \gamma^{-1}d\gamma)$, we realize immediately that the gauge transform by g of the connection ω and its curvature f must have the form

$$ {}^g\omega = g(\omega - \sigma_g) \tag{3.4.1} $$

and

$$ {}^g f = g(f). \tag{3.4.2} $$

If ϕ is a field, the gauge transform of ϕ under a standard gauge transformation $\gamma \in \mathrm{Map}(M, G)$ is given by ${}^\gamma\phi = \gamma\phi\gamma^{-1}$ and that of its covariant derivative $D\phi$ by ${}^\gamma D\phi = \gamma D\phi\gamma^{-1}$, as is well–known. These relations generalize immediately to any gauge transformation $g \in \mathrm{Gau}(M, \mathfrak{g})$, yielding

$$ {}^g\phi = g(\phi) \tag{3.4.3} $$

and likewise

$$ {}^g D\phi = g(D\phi) \tag{3.4.4} $$

[12]. For a dual field υ, we have similarly

$$^{g}\upsilon = g^{\vee}(\upsilon) \tag{3.4.5}$$

and

$$^{g}D\upsilon = g^{\vee}(D\upsilon). \tag{3.4.6}$$

Gauge transformation of connections and (dual) fields constitutes a left action of the gauge transformation group $\mathrm{Gau}(M, \mathfrak{g})$ on fields, that is, for any two gauge transformations $g, h \in \mathrm{Gau}(M, \mathfrak{g})$, ${}^{g \diamond h}\mathcal{F} = {}^{g}({}^{h}\mathcal{F})$, where $\mathcal{F} = \omega$, ϕ, υ, f, $D\phi$, $D\upsilon$ is anyone of the fields considered above.

Next, we shift to 2–term L_∞ algebra gauge theory and look for a sensible definition of gauge transformation action in this higher context extending without trivializing the ordinary gauge transformation action as formulated above. But, before doing that, a preliminary issue must be settled. 2–term L_∞ gauge transformations form a strict 2–group $\mathrm{Gau}(M, \mathfrak{v})$ comprising 1– and 2–gauge transformations (cf. subsect. 3.3). The natural question arises about whether one or both types of gauge transformations act on fields. From our analysis of $\mathrm{Gau}(M, \mathfrak{v})$, it emerges that it is the 1–gauge transformations which answer to the gauge transformations of ordinary gauge theory, while 2–gauge transformations constitute what may be called gauge for gauge transformations. It is thus natural to assume that only 1–gauge transformations act effectively on fields. So, we shall restrict to the 1–cell set $\mathrm{Gau}_1(M, \mathfrak{v})$ of $\mathrm{Gau}(M, \mathfrak{v})$, which, we recall, is an ordinary group. The role of the 2–cell set $\mathrm{Gau}_2(M, \mathfrak{v})$ of $\mathrm{Gau}(M, \mathfrak{v})$ will become clear in the analysis of the global properties of the theory.

In ordinary gauge theory, covariant differentiation is gauge covariant. This is often stated by saying that for any gauge transformation g and any field $\mathcal{F}$, ${}^{g}D\mathcal{F}$

[12] For any field $\mathcal{F}$, ${}^{g}D\mathcal{F} \equiv {}^{g}(D\mathcal{F})$ is defined by replacing each occurrence of each field $\mathcal{G}$ in the expression of $D\mathcal{F}$ by ${}^{g}\mathcal{G}$.

has the same form as ${}^g\mathcal{F}$. An equivalent way of telling this is that ${}^gD\mathcal{F}$ is given by formal covariant differentiation of ${}^g\mathcal{F}$ treating g and σ_g as if they were formally covariantly constant, a property ultimately guaranteed by (3.3.1), (3.3.2). It is reasonable to require that covariant differentiation in 2–term L_∞ algebra gauge theory has the same basic feature. So, the condition determining the form of ${}^g\mathcal{F}$ in 2–term L_∞ algebra gauge theory is that, for any gauge transformation g, ${}^gD\mathcal{F}$ is given by formal covariant differentiation of ${}^g\mathcal{F}$ treating the components g_0, g_1, g_2 of g as well as σ_g, $\varSigma_g$, τ_g as if they were formally covariantly constant. Eventually, this property will ensue from the basic identities (3.3.3)–(3.3.5). Proceeding in this way, we obtain the expressions of the gauge transforms of the various types of fields reported below. As it turns out, the fact that covariant differentiation mixes fields of the same doublet and depends on an assigned connection doublet makes ${}^g\mathcal{F}$ depend in general on $\mathcal{F}$ and its doublet partner as well as the connection doublet, a property which has no analog in ordinary gauge theory.

Let $g \in \mathrm{Gau}_1(M, \mathfrak{v})$ be a 1–gauge transformation.

We consider first a connection doublet $(\omega, \varOmega)$. The gauge transformed connection doublet $({}^g\omega, {}^g\varOmega)$ is defined to be

$$ {}^g\omega = g_0(\omega - \sigma_g), \tag{3.4.7a} $$

$$ {}^g\varOmega = g_1(\varOmega - \varSigma_g + \tau_g(\omega - \sigma_g)) - \frac{1}{2} g_2(\omega - \sigma_g, \omega - \sigma_g). \tag{3.4.7b} $$

We verify that this prescription satisfies the requirements established above, by computing the gauge transformed curvature doublet $({}^gf, {}^gF)$ and checking that $({}^gf, {}^gF)$ is given by formal covariant differentiation of $({}^g\omega, {}^g\varOmega)$ with g treated as covariantly constant [13]. Indeed, substituting (3.4.7) into (3.2.11), we find

$$ {}^gf = g_0(f), \tag{3.4.8a} $$

[13] Here, the covariant derivative of a connection is taken conventionally to be its curvature.

$$ {}^gF = g_1(F - \tau_g(f)) + g_2(\omega - \sigma_g, f). \tag{3.4.8b} $$

Let a connection doublet (ω, Ω) be fixed. A bidegree (p,q) field doublet (ϕ, Φ) is said *canonical*, if the gauge transformed field doublet $({}^g\phi, {}^g\Phi)$ is given by

$$ {}^g\phi = g_0(\phi), \tag{3.4.9a} $$

$$ {}^g\Phi = g_1(\Phi - (-1)^{p+q}\tau_g(\phi)) + (-1)^{p+q}g_2(\omega - \sigma_g, \phi). \tag{3.4.9b} $$

Again, to see that this prescription has the properties required above, we compute the gauge transformed covariant derivative field doublet $({}^gD\phi, {}^gD\Phi)$ and check that $({}^gD\phi, {}^gD\Phi)$ is given by formal covariant differentiation of $({}^g\phi, {}^g\Phi)$ with g assumed covariantly constant. Substituting (3.4.7), (3.4.9) into (3.2.13) and using (3.2.11), we find indeed

$$ {}^gD\phi = g_0(D\phi), \tag{3.4.10a} $$

$$ \begin{aligned} {}^gD\Phi = g_1(D\Phi + (-1)^{p+q}\tau_g(D\phi)) \\ - (-1)^{p+q}g_2(\omega - \sigma_g, D\phi) + (-1)^{p+q}g_2(f, \phi). \end{aligned} \tag{3.4.10b} $$

Note that the gauge transformation action depends explicitly on ω, as predicted earlier. Note also that the field doublet $(D\phi, D\Phi)$ is not canonical: (3.4.10b) cannot be recovered from (3.4.9b) just by replacing (ϕ, Φ) with $(D\phi, D\Phi)$ and shifting p into $p+1$ because of the extra f dependent term in the right hand side. This is an unavoidable consequence of explicit ω dependence. Finally, we observe that the curvature doublet (f, F) is a bidegree $(2,0)$ canonical field doublet.

Similarly, a bidegree (r,s) dual field doublet (Υ, υ) is said canonical, if the gauge transformed dual field doublet $({}^g\Upsilon, {}^g\upsilon)$ is given by

$$ {}^g\Upsilon = {g^\vee}_1(\Upsilon), \tag{3.4.11a} $$

$$ {}^g\upsilon = {g^\vee}_0(\upsilon - (-1)^{r+s}{\tau_g}^\vee(\Upsilon)) - (-1)^{r+s}{g^\vee}_2(g_0(\omega - \sigma_g), \Upsilon), \tag{3.4.11b} $$

where ${\tau_g}^\vee$ is defined by the relation $\langle \Xi, \tau_g(x)\rangle = \langle {\tau_g}^\vee(\Xi), x\rangle$. The gauge trans-

formed covariant derivative dual field doublet $({}^g D\varUpsilon, {}^g Dv)$ can be obtained readily by substituting (3.4.7), (3.4.11) into (3.2.15) and using (3.2.11),

$$ {}^g D\varUpsilon = g^{\vee}{}_1(D\varUpsilon), \tag{3.4.12a} $$

$$ {}^g Dv = g^{\vee}{}_0(Dv + (-1)^{r+s}\tau_g{}^{\vee}(D\varUpsilon)) \tag{3.4.12b} $$
$$ + (-1)^{r+s} g^{\vee}{}_2(g_0(\omega - \sigma_g), D\varUpsilon) - (-1)^{r+s} g^{\vee}{}_2(g_0(f), \varUpsilon) $$

and, as before, has the required properties. Again, the gauge transformation action is explicitly ω dependent and $(D\varUpsilon, Dv)$ is not canonical.

Gauge transformation preserves the field/dual field doublet pairing:

$$ \langle({}^g\varUpsilon, {}^g v), ({}^g\phi, {}^g\varPhi)\rangle = \langle(\varUpsilon, v), (\phi, \varPhi)\rangle, \tag{3.4.13} $$

a simple consequence of (3.2.19), (3.4.9), (3.4.11).

By (3.4.8a), the 2–form curvature f has this remarkable property: $f = 0$ implies that ${}^g f = 0$ for all 1–gauge transformations $g \in \mathrm{Gau}_1(M, \mathfrak{v})$. Thus, the vanishing 2–form curvature condition

$$ f = 0 \tag{3.4.14} $$

can be imposed consistently with gauge covariance. Indeed, it is rather natural to do so, as is immediate to see. By (3.4.8b), when (3.4.14) holds, the 3–form curvature gauge transforms very simply as

$$ {}^g F = g_1(F). \tag{3.4.15} $$

Further, by (3.4.10), if $(\phi, \varPhi)$ is a canonical field doublet, then also $(D\phi, D\varPhi)$ is and similarly, by (3.4.12), for a canonical dual field doublet $(\varUpsilon, v)$. In fact, condition (3.4.14) is closely related to the so called "vanishing fake curvature condition" arising in other formulations of higher gauge theory with strict structure 2–group V [25, 30, 32, 33]. Such condition guarantees that 2–parallel transport is a 2–functor from the path 2–groupoid $P_2(M)$ of M to the delooping 2–groupoid BV

of V generalizing the analogous property of parallel transport in standard gauge theory. To know whether such condition arises naturally also in our formulation of semistrict higher gauge theory, one would need a suitable definition of parallel transport. But, as we explained at the beginning of this section, our approach, relying on the automorphism 2–group $\mathrm{Aut}(\mathfrak{v})$ of a structure Lie 2–algebra $\mathfrak{v}$ rather than a structure 2–group V, is apparently unsuitable for the treatment of parallel transport. The issue is thus still open. More on this in subsect. 3.9.

Gauge transformation of connection and canonical field/dual field doublets constitutes a left action of the 1–gauge transformation group $\mathrm{Gau}_1(M, \mathfrak{v})$ on fields. Indeed, for any two 1–gauge transformations $g, h \in \mathrm{Gau}_1(M, \mathfrak{v})$, ${}^{g\diamond h}\mathcal{F} = {}^{g}({}^{h}\mathcal{F})$, where $\mathcal{F} = \omega$, $\varOmega$, ϕ, $\varPhi$, $\varUpsilon$, υ, f, F, $D\phi$, $D\varPhi$, $D\varUpsilon$, $D\upsilon$ is anyone of the fields considered above, as can be verified straightforwardly from (3.4.7)–(3.4.12) using (3.3.8a)–(3.3.8d) systematically. This is a very basic property and its holding indicates that our definition of gauge transformation is sound.

Later, we shall encounter other more complicated forms of gauge transformations action involving combinations of a large number of fields. The one presented above is however canonical in many ways.

3.5 2–term L_∞ algebra gauge rectifier

In the previous subsection, we have found that gauge transformation acts linearly on the components of canonical field doublets (cf. eqs. (3.4.9)). The transformation however mixes the components and further depends on a preassigned connection doublet. The gauge transformation action on canonical dual field doublets has analogous features (cf. eqs. (3.4.11)). The complicated way these doublets behave under gauge transformation makes it difficult to control gauge covariance and turns out to be a major obstacle to consistently carrying out gauge fixing in semistrict higher gauge theory. Fortunately, using gauge rectifiers,

it is possible to perform field redefinitions turning canonical (dual) field doublets into rectified ones transforming linearly, with no mixing and independently from any connection doublet under the gauge transformation action.

A field doublet $(\phi, \varPhi)$ is said *rectified*, if, under any 1–gauge transformation $g \in \mathrm{Gau}_1(M, \mathfrak{v})$, it transforms as

$$ {}^g\phi = g_0(\phi), \tag{3.5.1a} $$

$$ {}^g\varPhi = g_1(\varPhi). \tag{3.5.1b} $$

Similarly, a dual field doublet $(\varUpsilon, \upsilon)$ is rectified, if, under g, it transforms as

$$ {}^g\varUpsilon = g^\vee{}_1(\varUpsilon), \tag{3.5.2a} $$

$$ {}^g\upsilon = g^\vee{}_0(\upsilon). \tag{3.5.2b} $$

By definition, then, the gauge transformation action on rectified (dual) field doublets is linear, free of component mixing and independent from any given connection doublet. These makes rectified doublets very convenient to handle in field theoretic applications.

Comparing eqs. (3.5.1), (3.4.9), it appears that a canonical field doublet $(\phi, \varPhi)$ is not rectified. Similarly, from inspecting (3.5.2), (3.4.11), we find that a canonical dual field doublet $(\varUpsilon, \upsilon)$ is not rectified either. Gauge rectifiers remedy this defect, as we now show.

We assume again that M is diffeomorphic to $\mathbb{R}^d$. A pair (λ, μ) with $\lambda \in \Omega^0(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0 \wedge \hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$, $\mu \in \Omega^1(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$, is a 2*–term* L_∞ *algebra gauge rectifier* if, under any 1–gauge transformation $g \in \mathrm{Gau}_1(M, \mathfrak{v})$, it transforms as follows

$$ {}^g\lambda(x, y) = g_1(\lambda(g^{-1}{}_0(x), g^{-1}{}_0(y)) + g^{-1}{}_2(x, y)), \tag{3.5.3a} $$

$$ {}^g\mu(x) = g_1(\mu(g^{-1}{}_0(x)) - \tau_g(g^{-1}{}_0(x)) - \lambda(\sigma_g, g^{-1}{}_0(x))). \tag{3.5.3b} $$

Gauge rectifiers span an affine space and, in this respect, they are akin to ordinary

gauge connections. It is immediately checked using (2.3.3c), (3.3.8b), (3.3.8d) that, for any two 1–gauge transformations $g, h \in \mathrm{Gau}_1(M, \mathfrak{v})$, ${}^{g\diamond h}\mathcal{F} = {}^{g}({}^{h}\mathcal{F})$, where $\mathcal{F} = \lambda, \mu$. Therefore, 1–gauge transformation of gauge rectifiers constitutes a left action of the group $\mathrm{Gau}_1(M, \mathfrak{v})$ on their space, as for fields. We have not been able to ascertain whether gauge rectifiers exist in general. We assume that they do in what follows.

Given $\lambda \in \mathrm{Hom}(\hat{\mathfrak{v}}_0 \wedge \hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1)$, let

$$[x, y]_\lambda = [x, y] - \partial\lambda(x, y), \tag{3.5.4a}$$

$$[x, X]_\lambda = [x, X] - \lambda(x, \partial X), \tag{3.5.4b}$$

$$\begin{aligned}[x, y, z]_\lambda = [x, y, z] &- [x, \lambda(y, z)]] - [y, \lambda(z, x)]] - [z, \lambda(x, y)] \\ &- \lambda(x, [y, z]) - \lambda(y, [z, x]) - \lambda(z, [x, y]) \\ &+ \lambda(x, \partial\lambda(y, z)) + \lambda(y, \partial\lambda(z, x)) + \lambda(z, \partial\lambda(x, y)).\end{aligned} \tag{3.5.4c}$$

Then, as is easily verified, $\mathfrak{v}_\lambda = (\mathfrak{v}_0, \mathfrak{v}_1, \partial, [\cdot, \cdot]_\lambda, [\cdot, \cdot, \cdot]_\lambda)$ is a 2–term L_∞ algebra, the λ–*deformation of* $\mathfrak{v}$. Thus, once a gauge rectifier (λ, μ) is assigned, we have a λ–deformation $\mathfrak{v}_\lambda$ of $\mathfrak{v}$ pointwise on M.

The λ–deformed brackets transform covariantly under 1–gauge transformation, that is, for $g \in \mathrm{Gau}_1(M, \mathfrak{v})$,

$$[x, y]_{{}^g\lambda} = g_0([{g^{-1}}_0(x), {g^{-1}}_0(y)]_\lambda), \tag{3.5.5a}$$

$$[x, X]_{{}^g\lambda} = g_1([{g^{-1}}_0(x), {g^{-1}}_1(X)]_\lambda), \tag{3.5.5b}$$

$$[x, y, z]_{{}^g\lambda} = g_1([{g^{-1}}_0(x), {g^{-1}}_0(y), {g^{-1}}_0(z)]_\lambda). \tag{3.5.5c}$$

This can be shown straightforwardly by combining (2.3.1b)–(2.3.1d) with (3.5.4). By this remarkable property, the deformed brackets are the truly natural ones under 1–gauge transformation.

With a gauge rectifier (λ, μ), there is associated a *derived rectifier* $(v_{\lambda,\mu}, w_{\lambda,\mu})$,

where $\upsilon_{\lambda,\mu} \in \Omega^1(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0 \wedge \hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$, $w_{\lambda,\mu} \in \Omega^2(M, \mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$ and

$$\upsilon_{\lambda,\mu}(x,y) = d\lambda(x,y) - \mu([x,y]_\lambda) + [x,\mu(y)]_\lambda - [y,\mu(x)]_\lambda, \tag{3.5.6a}$$

$$w_{\lambda,\mu}(x) = d\mu(x) + \mu(\partial\mu(x)). \tag{3.5.6b}$$

Under any 1–gauge transformation $g \in \mathrm{Gau}_1(M, \mathfrak{v})$, we have

$$\upsilon_{{}^g\lambda,{}^g\mu}(x,y) = g_1(\upsilon_{\lambda,\mu}({g^{-1}}_0(x), {g^{-1}}_0(y)) - [\sigma_g, {g^{-1}}_0(x), {g^{-1}}_0(y)]_\lambda), \tag{3.5.7a}$$

$$\begin{aligned} w_{{}^g\lambda,{}^g\mu}(x) = g_1\Big(& w_{\lambda,\mu}({g^{-1}}_0(x)) - \upsilon_{\lambda,\mu}(\sigma_g, {g^{-1}}_0(x)) \\ & - [{g^{-1}}_0(x), \varSigma_g - \frac{1}{2}\lambda(\sigma_g,\sigma_g) + \mu(\sigma_g)]_\lambda + \frac{1}{2}[\sigma_g,\sigma_g,{g^{-1}}_0(x)]_\lambda\Big). \end{aligned} \tag{3.5.7b}$$

These relations follow from writing g_2 and τ_g in terms of λ, μ via (3.5.3) and substituting the resulting expressions in (3.3.5c) and (3.3.4). Though this is not immediately evident, $\upsilon_{\lambda,\mu}$, $w_{\lambda,\mu}$ are natural differential expressions in λ, μ which appear repeatedly as building blocks of 1–gauge covariant expressions.

We now show how one can turn canonical (dual) field doublets into rectified ones using a chosen 2–term L_∞ algebra gauge rectifier (λ, μ). Suppose that a connection doublet $(\omega, \varOmega)$ is given. Let $(\phi, \varPhi)$ be a bidegree (p,q) canonical field doublet. Then, naturally associated with $(\phi, \varPhi)$, there is a rectified field doublet $(\phi_{\lambda,\mu}, \varPhi_{\lambda,\mu})$, where $\phi_{\lambda,\mu} = \phi$ and

$$\varPhi_{\lambda,\mu} = \varPhi + (-1)^{p+q}\lambda(\omega,\phi) - (-1)^{p+q}\mu(\phi). \tag{3.5.8}$$

Similarly, a bidegree (r,s) canonical dual field doublet $(\varUpsilon, \upsilon)$ yields naturally a rectified dual field doublet $(\varUpsilon_{\lambda,\mu}, \upsilon_{\lambda,\mu})$, where $\varUpsilon_{\lambda,\mu} = \varUpsilon$ and

$$\upsilon_{\lambda,\mu} = \upsilon - (-1)^{r+s}\lambda^\vee(\omega,\varUpsilon) + (-1)^{r+s}\mu^\vee(\varUpsilon) \tag{3.5.9}$$

and the gauge corectifier $(\lambda^\vee, \mu^\vee)$ is defined by $\langle\lambda^\vee(x,\varXi), y\rangle = -\langle\varXi, \lambda(x,y)\rangle$, $\langle\mu^\vee(\varXi), x\rangle = -\langle\varXi, \mu(x)\rangle$.

Given a connection doublet $(\omega, \varOmega)$ and a gauge rectifier (λ, μ), it is possible to define a rectified covariant derivative $D_{\lambda,\mu}$ mapping rectified (dual) field doublets

into rectified ones. For a bidegree (p,q) rectified field doublet $(\phi,\varPhi)$, the bidegree $(p+1,q)$ rectified covariant derivative field doublet $(D_{\lambda,\mu}\phi, D_{\lambda,\mu}\varPhi)$ is given by

$$D_{\lambda,\mu}\phi = d\phi + [\omega,\phi]_\lambda + \partial\mu(\phi), \tag{3.5.10a}$$

$$D_{\lambda,\mu}\varPhi = d\varPhi + [\omega,\varPhi]_\lambda + \mu(\partial\varPhi). \tag{3.5.10b}$$

Similarly, for a bidegree (r,s) rectified dual field doublet $(\varUpsilon,\upsilon)$, the bidegree $(r+1,s)$ rectified covariant derivative dual field doublet $(D_{\lambda,\mu}\varUpsilon, D_{\lambda,\mu}\upsilon)$ is

$$D_{\lambda,\mu}\varUpsilon = d\varUpsilon + [\omega,\varUpsilon]_\lambda{}^\vee + \partial^\vee\mu^\vee(\varUpsilon), \tag{3.5.11a}$$

$$D_{\lambda,\mu}\upsilon = d\upsilon + [\omega,\upsilon]_\lambda{}^\vee + \mu^\vee(\partial^\vee\upsilon), \tag{3.5.11b}$$

where the λ–deformed cobracket $[\cdot,\cdot]_\lambda{}^\vee$ is defined in the same way as the undeformed one (cf. eqs. (3.2.16b), (3.2.16c)). Rectified covariant differentiation intertwines naturally with the duality pairing,

$$d\langle\upsilon,\phi\rangle = \langle D_{\lambda,\mu}\upsilon,\phi\rangle - (-1)^{r+s}\langle\upsilon, D_{\lambda,\mu}\phi\rangle, \tag{3.5.12a}$$

$$d\langle\varUpsilon,\varPhi\rangle = \langle D_{\lambda,\mu}\varUpsilon,\varPhi\rangle + (-1)^{r+s}\langle\varUpsilon, D_{\lambda,\mu}\varPhi\rangle. \tag{3.5.12b}$$

Compare with (3.2.20).

3.6 The 2–group of crossed module gauge transformations

We are going to define a notion of gauge transformation naturally hinged on an arbitrary Lie crossed module (G,H,t,m). We than shall show that these crossed module gauge transformations form naturally a strict 2–group. We consider again the case where M is diffeomorphic to $\mathbb{R}^d$.

A *crossed module* 1*–gauge transformation* consists of the following data:

1. a map $\gamma \in \mathrm{Map}(M,G)$;

2. an element $\chi_\gamma \in \Omega^1(M,\mathfrak{h})$.

In the following, we are going to denote a crossed module 1–gauge transformation such as the above as (γ, χ_γ) or simply as γ. Again, as before, in so doing, we are not implying that χ_γ is determined by γ, but only that its the partner of γ in the gauge transformation. We shall denote the set of crossed 1–gauge transformation by $\mathrm{Gau}_1(M, G, H)$. The notion of gauge transformation defined here coincides with that given in refs. [32, 33].

For any two crossed module 1–gauge transformations $\gamma, \beta \in \mathrm{Gau}_1(M, G, H)$, a *crossed module 2–gauge transformation* from γ to β consists of the following data:

1. a map $\Theta \in \mathrm{Map}(M, H)$;

2. an element $B_\Theta \in \Omega^1(M, \mathfrak{h})$.

They are required to satisfy the following relations,

$$\beta = t(\Theta)\gamma, \tag{3.6.1a}$$

$$\chi_\gamma - \chi_\beta = B_\Theta. \tag{3.6.1b}$$

We shall denote a crossed module 2–gauge transformation like this one as (Θ, B_Θ) or simply as Θ, meaning as usual in the former case that B_Θ is the partner of Θ. We shall also write $\Theta : \gamma \Rightarrow \beta$ to emphasize its source and target. We shall denote the set of all crossed module 2–gauge transformations $\Theta : \gamma \Rightarrow \beta$ by $\mathrm{Gau}_2(M, G, H)(\gamma, \beta)$ and the set of all 2–gauge transformations Θ by $\mathrm{Gau}_2(M, G, H)$. Note that, by (3.6.1b), the datum B_Θ is determined γ, β and, so, is essentially redundant.

Next, we shall show that it is possible to define a composition and an inversion law and a unit in $\mathrm{Gau}_1(M, G, H)$ and horizontal and vertical composition and inversion laws and units in $\mathrm{Gau}_2(M, G, H)$, making $(\mathrm{Gau}_1(M, G, H), \mathrm{Gau}_2(M, G, H))$ a strict 2–group (cf. subsect. 2.7).

The composition and inversion laws and the unit of 1–gauge transformation are defined by the relations

$$\beta \diamond \gamma = \beta\gamma, \tag{3.6.2a}$$

$$\chi_{\beta \diamond \gamma} = \chi_\beta + \dot{m}(\beta)(\chi_\gamma), \tag{3.6.2b}$$

$$\gamma^{-1_\diamond} = \gamma^{-1}, \tag{3.6.2c}$$

$$\chi_{\gamma^{-1_\diamond}} = -\dot{m}(\gamma^{-1})(\chi_\gamma), \tag{3.6.2d}$$

$$\iota = 1_G, \tag{3.6.2e}$$

$$\chi_\iota = 0. \tag{3.6.2f}$$

where $\gamma, \beta \in \mathrm{Gau}_1(M, G, H)$ and $\dot{m}$ is defined in (2.8.2). In (3.6.2a), (3.6.2c), (3.6.2e), the composition, inversion and unit in the right hand side are those of G thought of as holding pointwise on M.

The horizontal and vertical composition and inversion laws and the units of 2–gauge transformations are defined by the relations

$$\Lambda \diamond \Theta = \Lambda m(\zeta)(\Theta), \tag{3.6.3a}$$

$$B_{\Lambda \diamond \Theta} = B_\Lambda + \dot{m}(\zeta)(B_\Theta) + Q(\zeta \dot{t}(\chi_\beta)\zeta^{-1}, \Lambda), \tag{3.6.3b}$$

$$\Theta^{-1_\diamond} = m(\gamma^{-1})(\Theta^{-1}), \tag{3.6.3c}$$

$$B_{\Theta^{-1_\diamond}} = -\dot{m}(\gamma^{-1})(B_\Theta) + \dot{m}(\gamma^{-1})(Q(t(\Theta)^{-1}\dot{t}(\chi_\beta)t(\Theta), \Theta)), \tag{3.6.3d}$$

$$\Pi \bullet \Xi = \Pi\Xi, \tag{3.6.3e}$$

$$B_{\Pi \bullet \Xi} = B_\Xi + B_\Pi, \tag{3.6.3f}$$

$$\Xi^{-1_\bullet} = \Xi^{-1}, \tag{3.6.3g}$$

$$B_{\Xi^{-1_\bullet}} = -B_\Xi, \tag{3.6.3h}$$

$$I_\gamma = 1_H, \tag{3.6.3i}$$

$$B_{I_\gamma} = 0, \tag{3.6.3j}$$

where $\gamma, \beta, \zeta, \eta \in \mathrm{Gau}_1(M, G, H)$ and $\Theta, \Lambda, \Xi, \Pi \in \mathrm{Gau}_2(M, G, H)$, with $\Theta : \gamma \Rightarrow \beta$, $\Lambda : \zeta \Rightarrow \eta$ and Ξ, Π composible and $\dot{t}$ and Q are defined in (2.8.1) and (2.9.8). In (3.6.3a), (3.6.3c), (3.6.3e), (3.6.3g), (3.6.3i), the composition, inversion and unit in the right hand side are those of H holding pointwise on M and similarly for the G–action m.

It is straightforward to verify that the composition, inversion and unit structures just defined satisfy the axioms (2.7.1), so that $(\mathrm{Gau}_1(M, G, H), \mathrm{Gau}_2(M, G, H))$ is indeed a strict 2–group, as claimed.

The pair $(\mathrm{Map}(M, G), \mathrm{Map}(M, H))$ has a structure of infinite dimensional Lie crossed module induced by that of (G, H) pointwise on M. $(\mathrm{Map}(M, G), \mathrm{Map}(M, H))$ in turn can be viewed as an infinite dimensional strict 2–group $\mathrm{Map}(M, V)$ using the conversion prescriptions listed in subsect. 2.7. Eqs. (3.6.2a), (3.6.2c), (3.6.2e), (3.6.3a), (3.6.3c), (3.6.3e), (3.6.3g), (3.6.3i) define precisely the 2–group operations of $\mathrm{Map}(M, V)$ expressed in terms of the crossed module structure of $(\mathrm{Map}(M, G), \mathrm{Map}(M, H))$.

Once the above is realized, inspection of (3.6.2) reveals that the 1–cell group $\mathrm{Gau}_1(M, G, H)$ is the semidirect product $\Omega^1(M, \mathfrak{h}) \rtimes \mathrm{Map}(M, V_1)$ associated with a certain right action of $\mathrm{Map}(M, V_1)$ on $\Omega^1(M, \mathfrak{h})$. Unlike the group of 2–term L_∞ algebra 1–gauge transformation $\mathrm{Gau}_1(M, \mathfrak{v})$ in subsect. 3.3, $\mathrm{Gau}_1(M, G, H)$ is not a proper subgroup of $\Omega^1(M, \mathfrak{h}) \rtimes \mathrm{Map}(M, V_1)$, since there are no differential relations obeyed by its elements analogous to (3.3.3)–(3.3.5). Again, this leads us to conjecture that the full 2–group $\mathrm{Gau}(M, G, H)$ may be described as a semidirect product 2–group defined by the relations (3.6.2), (3.6.3), for a suitable definition of the latter notion.

There exists a natural strict 2–group 1–morphism (cf. sect. 2.7, eq. (2.7.3)) from the crossed module gauge transformation 2–group $\mathrm{Gau}(M, G, H)$ to the gauge transformation 2–group $\mathrm{Gau}(M, \mathfrak{v})$, where $\mathfrak{v}$ is the strict 2–term L_∞ algebra corresponding to the differential Lie crossed module $(\mathfrak{g}, \mathfrak{h})$. The morphism is

defined by following expressions

$$g_\gamma = \phi_\gamma, \tag{3.6.4a}$$

$$\sigma_{g_\gamma} = \gamma^{-1}d\gamma + \gamma^{-1}\dot{t}(\chi_\gamma)\gamma, \tag{3.6.4b}$$

$$\Sigma_{g_\gamma} = \dot{m}(\gamma^{-1})\Big(d\chi_\gamma + \frac{1}{2}[\chi_\gamma, \chi_\gamma]\Big), \tag{3.6.4c}$$

$$\tau_{g_\gamma}(x) = \widehat{m}(x)(\dot{m}(\gamma^{-1})(\chi_\gamma)), \tag{3.6.4d}$$

$$F_\varLambda(x) = \varPhi_{\zeta,\varLambda}(x), \tag{3.6.4e}$$

$$A_{F_\varLambda} = \dot{m}(\zeta^{-1})(-\varLambda^{-1}d\varLambda + \chi_\zeta + \varLambda^{-1}(B_\varLambda - \chi_\zeta)\varLambda), \tag{3.6.4f}$$

for $\gamma, \zeta, \eta \in \mathrm{Gau}_1(M, G, H)$ and $\varLambda \in \mathrm{Gau}_2(M, G, H)$ with $\varLambda : \zeta \Rightarrow \eta$, where the right hand sides of (3.6.4a) and (3.6.4e) are defined by (2.9.6) and (2.9.7) pointwise on M and $\widehat{m}$ is defined in (2.8.3).

The mappings $\gamma \to \phi_\gamma$ and $(\zeta, \varLambda) \to \varPhi_{\zeta,\varLambda}$ define a strict 2–group 1–morphim from $\mathrm{Map}(M, V)$ to $\mathrm{Map}(M, \mathrm{Aut}(\mathfrak{v}))$ which is the pointwise version of the "adjoint" 2–group 1–morphim from V to $\mathrm{Aut}(\mathfrak{v})$ defined in sect. 2.9. (3.6.4) above extend such morphism to one from $\mathrm{Gau}(M, G, H)$ to $\mathrm{Gau}(M, \mathfrak{v})$.

Let $(\omega, \varOmega)$ be a connection doublet. Inserting eqs. (3.6.4b)–(3.6.4d) into the relations (3.4.7), we obtain

$${}^{g_\gamma}\omega = \gamma\omega\gamma^{-1} - d\gamma\gamma^{-1} - \dot{t}(\chi_\gamma), \tag{3.6.5a}$$

$${}^{g_\gamma}\varOmega = \dot{m}(\gamma)(\varOmega) - \widehat{m}(\gamma\omega\gamma^{-1} - d\gamma\gamma^{-1} - \dot{t}(\chi_\gamma))(\chi_\gamma) - d\chi_\gamma - \frac{1}{2}[\chi_\gamma, \chi_\gamma]. \tag{3.6.5b}$$

Inserting eqs. (3.6.4b)–(3.6.4d) into (3.4.8), we find further

$${}^{g_\gamma}f = \gamma f\gamma^{-1}, \tag{3.6.6a}$$

$${}^{g_\gamma}F = \dot{m}(\gamma)(F) - \widehat{m}(\gamma f\gamma^{-1})(\chi_\gamma). \tag{3.6.6b}$$

Remarkably, these expressions are identical to those obtained originally in refs. [32,33]. This shows that (2.8.3), (3.4.8) provide the appropriate generalization of the gauge transformation action for a 2–term L_∞ gauge theory.

3.7 Review of principal 2–bundle theory

The analysis of the global aspects in gauge theory consists in determining how locally defined data glue in a globally consistent manner. In the same way as the global properties of ordinary gauge theory are described by the theory of principal bundles, those of higher gauge theory are expressed by the theory of principal 2–bundles. So, it is appropriate at this point to review these topics, recalling well–known basic facts of differential topology of principal bundles and then showing how these generalize to principal 2–bundles. Our presentation has no pretense of completeness or mathematical rigour and serves only the purpose of setting the terminology and the notation used later.

The definition of local data requires the choice of an *open covering* $U = \{U_i\}$ of M, that is a collection of open subsets $U_i \subset M$ such that

$$M = \bigcup\nolimits_i U_i. \tag{3.7.1}$$

The covering U is characterized by its *nerve*, which is the the collection of all non empty intersections $U_{i_0 \ldots i_n} = U_{i_0} \cap \ldots \cap U_{i_n} \neq \emptyset$ with $n \geq 0$.

Let G be a Lie group. We define a groupoid $\check{P}(U, G)$ as follows.

1. A 0–cell of $\check{P}(U, G)$ is collection $g = \{g_{ij}\}$ of smooth maps $g_{ij} \in \mathrm{Map}(U_{ij}, G)$ satisfying the condition

$$g_{ik} = g_{ij} g_{jk}, \qquad \text{on } U_{ijk}. \tag{3.7.2}$$

2. For any two 0–cell g, g' of $\check{P}(U, G)$, a 1–cell $g \to g'$ is a collection $\{h_i\}$ of smooth maps $h_i \in \mathrm{Map}(U_i, G)$ such that

$$g_{ij} h_j = h_i g'_{ij}, \qquad \text{on } U_{ij}. \tag{3.7.3}$$

3. For any 0–cell g, the identity 1–cell id_g of g is the collection $\{1_{Gi}\}$ of constant maps of $\mathrm{Map}(U_i, G)$ with value 1_G.

4. For any 1–cell $h : g \to g'$, the inverse 1–cell $h^{-1} : g' \to g$ is defined by

$$h^{-1} = \{h_i{}^{-1}\}. \tag{3.7.4}$$

5. For any two 1–cells $h : g \to g'$, $h' : g' \to g''$, the composition 1–cell $h'h : g \to g''$ is defined by

$$h'h = \{h_i h'_i\}. \tag{3.7.5}$$

It is straightforward to check that $\check{P}(U, G)$ is indeed a groupoid as anticipated.

The set of isomorphism classes of 0–cells g is nothing but the 1st Čech cohomology $\check{H}^1(U, G)$ of the covering U with coefficients in G. The dependence on U can be eliminated by switching to 1st Čech cohomology $\check{H}^1(M, G)$ of M with coefficients in G, which is defined as the direct limit under covering refinement of the cohomology $\check{H}^1(U, G)$,

$$\check{H}^1(M, G) = \lim_{\overrightarrow{U}} \check{H}^1(U, G). \tag{3.7.6}$$

In differential topology, $\check{H}^1(M, G)$ has a well-known interpretation: it is the set of isomorphism classes of smooth principal G–bundles P.

For any two 0–cells g, g', consider the set $\check{H}^2(U, G; g, g')$ of 1–cells $g \to g'$. If it is non empty, $\check{H}^2(U, G; g, g')$ depends only to the common isomorphism class of g and g' in $\check{H}^1(U, G)$ up to bijection, so that we can set $g' = g$ right away. The dependence on U is eliminated by switching 2nd Čech cohomology $\check{H}^2(P)$

$$\check{H}^2(P) = \lim_{\overrightarrow{U}} \check{H}^2(U, G; g, g), \tag{3.7.7}$$

where P is a principal G–bundles in the isomorphism class associated with g. In differential topology, $\check{H}^2(P)$ has also a well-known interpretation. If P is represented by a 0–cell g, $\check{H}^2(P)$ is represented by the group of 1–cells $g \to g$. $\check{H}^2(P)$ is thus the group of automorphisms of P, i. e. the gauge group $\mathrm{Gau}(P)$.

Let V be a strict Lie 2–group. Isomorphism classes of principal V–2–bundle P, the gauge group of one such bundle P and other appended structures can be characterized in a way which is a direct generalization of that of ordinary principal bundle theory illustrated above. We shall now go through this more explicitly following loosely the treatment of 2–bundles of refs. [32, 33], to which we refer the reader. (See also [66] for a comparison of different approaches.)

We recall first the definition of strict 2–groupoid. A *strict* 2*–groupoid* consists of the following set of data:

1. a set of 0-cells V_0;
2. for each pair of 0–cells x, y, a set of 1-cells $V_1(x, y)$;
3. for each triple of 0–cells x, y, z, a composition law of 1–cells $\circ : V_1(x, y) \times V_1(y, z) \to V_1(x, z)$;
4. for each pair of 0–cells x, y, a inversion law of 1–cells ${}^{-1\circ} : V_1(x, y) \to V_1(y, z)$;
5. for each 0–cell x, a distinguished unit 1–cell $1_x \in V_1(x, x)$;
6. for each pair of 0–cells x, y and for each pair of 1–cells $a, b \in V_1(x, y)$, a set of 2–cells $V_2(a, b)$;
7. for each triple of 0–cells x, y, z and for each pair of 1–cells $a, c \in V_1(x, y)$ and for each pair of 1–cells $b, d \in V_1(y, z)$, a horizontal composition law of 2–cells $\circ : V_2(a, c) \times V_2(b, d) \to V_2(b \circ a, d \circ c)$;
8. for each pair of 0–cells x, y and for each pair of 1–cells $a, b \in V_1(x, y)$, a horizontal inversion law of 2–cells ${}^{-1\circ} : V_2(a, b) \to V_2(a^{-1\circ}, b^{-1\circ})$;
9. for each pair of 0–cells x, y and for each triple of 1–cells $a, b, c \in V_1(x, y)$, a vertical composition law of 2–cells $\cdot : V_2(a, b) \times V_2(b, c) \to V_2(a, c)$;

10. for each pair of 0–cells x, y and for each pair of 1–cells $a, b \in V_1(x, y)$, a vertical inversion law of 2–cells ${}^{-1\cdot} : V_2(a, b) \to V_2(b, a)$;

11. for each pair of 0–cells x, y and for each 1–cell $a \in V_1(x, y)$, a distinguished unit 2–cell $1_a \in V_2(a, a)$.

These are required to satisfy the following axioms.

$$(c \circ b) \circ a = c \circ (b \circ a), \tag{3.7.8a}$$

$$a^{-1\circ} \circ a = 1_x, \qquad a \circ a^{-1\circ} = 1_y, \tag{3.7.8b}$$

$$a \circ 1_x = 1_y \circ a = a, \tag{3.7.8c}$$

$$(C \circ B) \circ A = C \circ (B \circ A), \tag{3.7.8d}$$

$$A^{-1\circ} \circ A = 1_{1_x} \qquad A \circ A^{-1\circ} = 1_{1_y}, \tag{3.7.8e}$$

$$A \circ 1_{1_x} = 1_{1_y} \circ A = A, \tag{3.7.8f}$$

$$(C \cdot B) \cdot A = C \cdot (B \cdot A), \tag{3.7.8g}$$

$$A^{-1\cdot} \cdot A = 1_a, \qquad A \cdot A^{-1\cdot} = 1_b, \tag{3.7.8h}$$

$$A \cdot 1_a = 1_b \cdot A = A, \tag{3.7.8i}$$

$$(D \cdot C) \circ (B \cdot A) = (D \circ B) \cdot (C \circ A). \tag{3.7.8j}$$

Here and in the following, $x, y, z, \cdots \in V_0$, $a, b, c, \cdots \in V_1$, $A, B, C, \cdots \in V_2$, where V_1 and V_2 denote the set of all 1– and 2–cells, respectively. For clarity, we often denote $a \in V_1(x, y)$ as $a : x \to y$ and $A \in V_2(a, b)$ as $A : a \Rightarrow b$. All identities involving the horizontal and vertical composition and inversion hold whenever defined. Relation (3.7.8j) is called again interchange law. In the following, we shall denote a 2–groupoid such as the above as V or (V_0, V_1, V_2) or $(V_0, V_1, V_2, \circ, {}^{-1\circ}, \cdot, {}^{-1\cdot}, 1_{-})$ to emphasize the underlying structure.

If $(V_0, V_1, V_2, \circ, {}^{-1\circ}, \cdot, {}^{-1\cdot}, 1_{-})$ is a strict 2–groupoid, then $(V_0, V_1, \circ, {}^{-1\circ}, 1_{-})$ and $(V_1, V_2, \cdot, {}^{-1\cdot}, 1_{-})$ are both groupoids. V can also be viewed as a strict 2–

category in which all 1–morphisms are invertible and all 2–morphisms are both horizontal and vertical invertible. A strict 2–group (V_1, V_2) is just a strict 2–groupoid (V_0, V_1, V_2) such that V_0 is the singleton set (cf. subsect. 2.7).

Let (G, H) be a Lie crossed module associated with the strict 2–group V. We define a strict 2–groupoid $\check{P}_2(U, G, H)$ as follows.

1. A 0–cell of $\check{P}_2(U, G, H)$ is a collection $(g, W) = \{g_{ij}, W_{ijk}\}$ of smooth maps $g_{ij} \in \mathrm{Map}(U_{ij}, G)$, $W_{ijk} \in \mathrm{Map}(U_{ijk}, H)$ satisfying the relations

$$g_{ij}g_{jk} = t(W_{ijk})g_{ik}, \qquad \text{on } U_{ijk}, \tag{3.7.9a}$$

$$m(g_{ij})(W_{jkl})W_{ijl} = W_{ijk}W_{ikl}, \qquad \text{on } U_{ijkl}. \tag{3.7.9b}$$

2. For any two 0–cells $(g, W), (g', W')$, a 1–cell $(g, W) \to (g', W')$ of $\check{P}_2(U, G, H)$ is a collection $(h, J) = \{h_i, J_{ij}\}$ of smooth maps $h_i \in \mathrm{Map}(U_i, G)$, $J_{ij} \in \mathrm{Map}(U_{ij}, H)$ satisfying the relations

$$h_i g'_{ij} = t(J_{ij})g_{ij}h_j, \qquad \text{on } U_{ij}, \tag{3.7.10a}$$

$$J_{ij}m(g_{ij})(J_{jk})W_{ijk} = m(h_i)(W'{}_{ijk})J_{ik}, \qquad \text{on } U_{ijk}. \tag{3.7.10b}$$

3. For any two 1–cells $(h, J), (h', J') : (g, W) \to (g', W')$, a 2–cell $(h, J) \Rightarrow (h', J')$ of $\check{P}_2(U, G, H)$ is a collection $K = \{K_i\}$ of smooth maps $K \in \mathrm{Map}(U_i, H)$ such that

$$h'{}_i = t(K_i)h_i, \tag{3.7.11a}$$

$$J'{}_{ij}m(g_{ij})(K_j) = K_iJ_{ij}. \tag{3.7.11b}$$

4. For any 0–cell (g, W), the identity 1–cell $\mathrm{id}_{(g,W)}$ of (g, W) is the collection $\{1_{Gi}, 1_{Hij}\}$ constant maps of $\mathrm{Map}(U_i, G)$ with value 1_G and $\mathrm{Map}(U_{ij}, H)$ with value 1_H.

5. For any 1–cell $(h, J) : (g, W) \to (g', W')$, the inverse 1–cell $(h, J)^{-1_\circ}$:

$(g', W') \to (g, W)$ is defined by

$$(h, J)^{-1_\circ} = \{{h_i}^{-1}, m({h_i}^{-1})({J_{ij}}^{-1})\}. \tag{3.7.12}$$

6. For any two 1–cells $(h, J) : (g, W) \to (g', W')$, $(h', J') : (g', W') \to (g'', W'')$, the composition 1–cell $(h', J') \circ (h, J) : (g, W) \to (g'', W'')$ is defined by

$$(h', J') \circ (h, J) = \{h_i {h'}_i, m(h_i)({J'}_{ij})J_{ij}\}. \tag{3.7.13}$$

7. For any 1–cell (h, J), the identity 2–cell $\mathrm{id}_{(h,J)}$ of (h, J) is the collection $\{1_{Hi}\}$ of constant maps $U_i \to H$ with value 1_H.

8. For any 2–cell $K : (h, J) \Rightarrow (h', J')$, the vertical inverse 2–cell $K^{-1_\cdot}$: $(h', J') \Rightarrow (h, J)$ is defined by

$$K^{-1_\cdot} = \{{K_i}^{-1}\}. \tag{3.7.14}$$

9. For any two 2–cells $K : (h, J) \Rightarrow (h', J')$, $K' : (h', J') \Rightarrow (h'', J'')$, the vertical composition 2–cell $K' \cdot K : (h, J) \Rightarrow (h'', J'')$ is defined by

$$K' \cdot K = \{{K'}_i K_i\}. \tag{3.7.15}$$

10. For any 2–cell $K : (h, J) \Rightarrow (h', J')$, the horizontal inverse 2–cell $K^{-1_\circ}$: $(h, J)^{-1_\circ} \Rightarrow (h', J')^{-1_\circ}$ is defined by

$$K^{-1_\circ} = \{m({h_i}^{-1})({K_i}^{-1})\} \tag{3.7.16}$$

11. For any two 2–cells $K : (h, J) \Rightarrow (h'', J'')$, $K' : (h', J') \Rightarrow (h''', J''')$ such that the composition 1–cells $(h', J') \circ (h, J)$, $(h''', J''') \circ (h'', J'')$ are defined the horizontal composition 2–cell $K' \circ K : (h', J') \circ (h, J) \Rightarrow (h''', J''') \circ (h'', J'')$ is defined by

$$K' \circ K = \{K_i m(h_i)({K'}_i)\}. \tag{3.7.17}$$

It is straightforward to check that $\check{P}_2(U, G, H)$ is indeed a strict 2–groupoid.

The set 1–isomorphisms classes of 0–cells (g, W) is the 1st Čech cohomology $\check{H}^1(U, G, H)$ of the covering U with coefficients in (G, H). Again, the dependence on U can be eliminated by switching to 1st Čech cohomology $\check{H}^1(M, G, H)$ of M with coefficients in (G, H), the direct limit under covering refinement of the cohomology $\check{H}^1(U, G, H)$,

$$\check{H}^1(M, G, H) = \lim_{\overrightarrow{U}} \check{H}^1(U, G, H). \tag{3.7.18}$$

By analogy to the theory of ordinary principal bundles, $\check{H}^1(M, G, H)$ is regarded as the set of isomorphism classes of smooth principal (G, H)–2–bundles P.

For any two 0–cells (g, W), (g', W'), let $\check{H}^2(U, G, H; (g, W), (g', W'))$ be the set of vertical 2–isomorphism classes of 1–cells $(g, W) \to (g', W')$. In case that it is non empty, $\check{H}^2(U, G, H; (g, W), (g', W'))$ depends only to the common 1–isomorphism class of (g, W) and (g', W') in $\check{H}^1(U, G, H)$ up to bijection so that we can set $(g', W') = (g, W)$ without loss of generality. Again, the dependence on U can be eliminated by switching to 2nd Čech cohomology $\check{H}^2(P)$ defined by

$$\check{H}^2(P) = \lim_{\overrightarrow{U}} \check{H}^2(U, G, H; (g, W), (g, W)), \tag{3.7.19}$$

where P is a principal (G, H)–2–bundles in the isomorphism class corresponding to (g, W). If P is represented by a 0–cell (g, W), then $\check{H}^2(P)$ is represented by the group of vertical 2–isomorphism classes of 1–cells $(g, W) \to (g, W)$. $\check{H}^2(P)$ is so the group of isomorphism classes of 1–automorphisms of P, which we can identify with the gauge group $\mathrm{Gau}(P)$ again by analogy with the theory of ordinary principal bundles.

The process does not stop here. For any two 1–cells $(h, J), (h', J') : (g, W) \to (g, W)$, consider the set $\check{H}^3(U, G, H; (h, J), (h', J'))$ of 2–cells $(h, J) \Rightarrow (h', J')$. If it is non empty, $\check{H}^3(U, G, H; (h, J), (h', J'))$ depends only to the common isomorphism class of (h, J), (h', J') in $\check{H}^2(U, G, H; (g, W), (g, W))$ up to bijection, so

that we can set $(h', J') = (h, J)$. Once more, the dependence on U is eliminated by switching 3rd Čech cohomology $\check{H}^2(\Gamma)$

$$\check{H}^3(P, \mathit{\Gamma}) = \lim_{\overrightarrow{U}} \check{H}^3(U, G, H; (h, J), (h, J)), \tag{3.7.20}$$

where P is a principal (G, H)–2–bundles in the isomorphism class corresponding to (g, W) and $\mathit{\Gamma}$ is an 1–automorphisms of P in the isomorphism class associated with (h, J). If P is represented by the 0–cell (g, W) and $\mathit{\Gamma}$ is represented by a 1–cell (h, J), then $\check{H}^3(P, \mathit{\Gamma})$ is represented by the vertical group of 2–cells $(h, J) \Rightarrow (h, J)$. $\check{H}^3(P, \mathit{\Gamma})$ is thus the group of 2–automorphisms of $\mathit{\Gamma}$, which we may view as the gauge for gauge group $\mathrm{Gau}(P, \mathit{\Gamma})$.

The description of principal 2–bundle we have formulated above looks more transparent though a bit more abstract when rephrased directly in terms of the strict Lie 2–group V underlying the Lie crossed module (G, H). Let us elaborate on this point.

Let G be a Lie group. As we have reviewed above, isomorphism classes of ordinary principal G–bundles are characterized in terms of gluing data $g = \{g_{ij}\}$ satisfying the condition (3.7.2) and determined up to an equivalence defined in terms of intertwiner data $h = \{h_i\}$ satisfying (3.7.3). The intuitive idea underlying the definition of principal 2–bundles is that of extending the Lie group G to a strict Lie 2–group $V = (V_1, V_2)$ with $G = V_1$ and "weakening" (3.7.2), (3.7.3) so that they hold only up to 2–cell errors $W = \{W_{ijk}\}$ and $J = \{J_{ij}\}$ drawn from V_2 and satisfying natural coherence conditions. Correspondingly, the groupoid $\check{P}(U, G)$ extends to a strict 2–groupoid $\check{P}_2(U, V)$. In Lie crossed module theoretic terms, $\check{P}_2(U, V)$ is just the 2–groupoid $\check{P}_2(U, G, H)$ we have studied above.

Explicitly, the content of $\check{P}_2(U, V)$ can be described as follows.

1. A 0–cell of $\check{P}_2(U, V)$ is a collection $(g, W) = \{g_{ij}, W_{ijk}\}$ of smooth maps $g_{ij} \in \mathrm{Map}(U_{ij}, V_1)$, $W_{ijk} \in \mathrm{Map}(U_{ijk}, V_2)$ such that

$$W_{ijk} : g_{ik} \Rightarrow g_{ij} \circ g_{jk}, \qquad \text{on } U_{ijk}, \tag{3.7.21a}$$

where $\circ$ denotes pointwise multiplication in V_1, satisfying in addition the coherence condition

$$(1_{g_{ij}} \circ W_{jkl}) \cdot W_{ijl} = (W_{ijk} \circ 1_{g_{kl}}) \cdot W_{ikl}, \qquad \text{on } U_{ijkl}, \tag{3.7.21b}$$

where $\circ$ and $\cdot$ denote pointwise horizontal and vertical multiplication in V_2, respectively. (3.7.21b) follows from equating the two pointwise 2–cells $g_{il} \Rightarrow g_{ij} \circ g_{jk} \circ g_{kl}$ that can be built using (3.7.21a).

2. For any two 0–cells (g, W), (g', W'), a 1–cell $(g, W) \to (g', W')$ of $\check{P}_2(U, V)$ is a collection $(h, J) = \{h_i, J_{ij}\}$ of smooth maps $h_i \in \mathrm{Map}(U_i, V_1)$, $J_{ij} \in \mathrm{Map}(U_{ij}, V_2)$ such that

$$J_{ij} : g_{ij} \circ h_j \Rightarrow h_i \circ g'_{ij}, \qquad \text{on } U_{ij}, \tag{3.7.22a}$$

satisfying the coherence condition

$$(J_{ij} \circ 1_{g'_{jk}}) \cdot (1_{g_{ij}} \circ J_{jk}) \cdot (W_{ijk} \circ 1_{h_k}) = (1_{h_i} \circ W'_{ijk}) \cdot J_{ik}, \qquad \text{on } U_{ijk} \tag{3.7.22b}$$

stemming from imposing that the two pointwise 2–cells $g_{ik} \circ h_k \Rightarrow h_i \circ g'_{ij} \circ g'_{jk}$ which can be built using (3.7.21a), (3.7.22a) are equal.

3. For any two 1–cells $(h, J), (h', J') : (g, W) \to (g', W')$, a 2–cell $(h, J) \Rightarrow (h', J')$ of $\check{P}_2(U, V)$ is a collection $K = \{K_i\}$ of smooth maps $K_i \in \mathrm{Map}(U_i, V_2)$ satisfying

$$K_i : h_i \Rightarrow h'_i, \qquad \text{on } U_i, \tag{3.7.23a}$$

and the coherence condition

$$J'_{ij} \cdot (1_{g_{ij}} \circ K_j) = (K_i \circ 1_{g'_{ij}}) \cdot J_{ij}, \qquad \text{on } U_{ij}, \tag{3.7.23b}$$

equating the two pointwise 2–cells $g_{ij} \circ h_j \Rightarrow h'_i \circ g'_{ij}$ built using (3.7.22a), (3.7.23a).

It is straightforward to check that, when written in (G, H) terms, (3.7.21)–(3.7.23)

precisely reproduce (3.7.9)–(3.7.11), respectively.

The description of principal bundles in terms of gluing data and their equivalence can be approached from alternative point of view advocated by Schreiber [36, 67], developing upon the results of refs. [32, 33]. We shall describe how the construction works for principal bundles and then we shall show how it generalizes to principal 2–bundles.

We begin by introducing the *Čech groupoid* $\check{C}(U)$ of the covering $U = \{U_i\}$. $\check{C}(U)$ is defined as follows.

1. A 0–cell of $\check{C}(U)$ is a pair (m, i) with $m \in U_i$.

2. A 1–cell of $\check{C}(U)$ is a triple (m, i, j) with $m \in U_{ij}$, constituting 1–arrows $(m, i) \to (m, j)$.

3. The identity $1_{(m,i)}$ of the 0–cell (m, i) is the 1–cell (m, i, i).

4. The composition $(m', j, k) \circ (m, i, j)$ of two 1–cells (m, i, j), (m', j, k) with $m = m'$ is the 1–cell (m, i, k).

5. The inverse $(m, i, j)^{-1\circ}$ of a 1–cells (m, i, j) is the 1–cell (m, j, i).

The sets of 0– and 1–cells of $\check{C}(U)$ are the disjoint unions $\coprod_i U_i$ and $\coprod_{ij} U_{ij}$, respectively, and so have a smooth structure induced by that of M, providing a smooth structure to $\check{C}(U)$.

Let G be a Lie group. The delooping groupoid BG of G is just G seen as the groupoid such that $BG_0 = \{*\}$, the singleton set, and $BG_1 = G$ with the smooth structure induced by that of G.

A 0–cell g of the groupoid $\check{P}(U, G)$ is equivalent to a smooth groupoid morphism $g : \check{C}(U) \to BG$ from the Čech groupoid $\check{C}(U)$ of U to the delooping groupoid BG of G. Indeed, a set of smooth gluing data $g = \{g_{ij}\}$ satisfying (3.7.2) defines a smooth functor from $\check{C}(U)$ to BG mapping the 1–cell (m, i, j)

to the 1–cell $g_{ji}(m)$. A 1–cell $h : g \to g'$ of the groupoid $\check{P}(U, G)$ is equivalent to a smooth natural isomorphism of the corresponding groupoid morphisms $g, g' : \check{C}(U) \to BG$. Indeed, a set of smooth intertwining data $h = \{h_i\}$ satisfying (3.7.3) defines an invertible smooth natural transformation of the functors g, g' mapping the 0–cell (m, i) to the 1–cell $h_i(m)$. In this way, a bijection is established between the isomorphism classes of smooth principal G–bundles P and the natural isomorphism classes of smooth functors $\check{C}(U) \to BG$. The theory of such principal bundle classes is therefore fully encoded the functor category $[\check{C}(U), BG]$.

The Čech 2–groupoid $\check{C}_2(U)$ of the covering U is the smooth strict 2–groupoid obtained by promoting the Čech groupoid $\check{C}(U)$ to a 2–groupoid by adding identity 2–cells.

Let V be a Lie 2–group. The delooping 2–groupoid BV of V is just V seen as the strict 2–groupoid such that $BV_0 = \{*\}$, the singleton set, $BV_1 = V_1$ and $BV_2 = V_2$ with the smooth structure induced by that of V.

A 0–cell (g, W) of the groupoid $\check{P}_2(U, V)$ is equivalent to a smooth 2–groupoid pseudomorphism $(g, W) : \check{C}_2(U) \to BV$ from the Čech 2–groupoid $\check{C}_2(U)$ of U to the delooping 2–groupoid BV of V. A 2–groupoid pseudomorphism is a pseudofunctor, that is an arrow $\Phi : A \to B$ of 2–categories associating to each 0– and each 1–cell of A a 0– and 1–cell of B, respectively, as a functor does, but preserving identity 1–cells and 1–cell compositions only up to 2–cells satisfying certain coherence conditions. A set of smooth data $\{g_{ij}, W_{ijk}\}$ satisfying (3.7.21) precisely defines a smooth pseudofunctor from $\check{C}_2(U)$ to BV mapping each 1–cell (m, i, j) to the 1–cell $g_{ji}(m)$ preserving the identities 1–cells (m, i, i) only up to the 2–cells $1_{g_{ii}{}^{-1}} \circ W_{iii} : 1 \Rightarrow g_{ii}$ and the 1–cell compositions $(m, i, k) = (m, j, k) \circ (m, i, j)$ only up the 2–cells $W_{kji} : g_{ki} \Rightarrow g_{kj} \circ g_{ji}$.

A 1–cell $(h, J) : (g, W) \to (g', W')$ of the groupoid $\check{P}_2(U, V)$ is equivalent to a smooth pseudonatural isomorphism of the corresponding 2–groupoid pseudo-

morphisms $(g, W), (g', W') : \check{C}_2(U) \to BV$. Given two arrows $\Phi, \Psi : A \to B$ of 2–categories, a pseudonatural isomorphism is a 2–arrow $\alpha : \Phi \Rightarrow \Psi$ associating to each object a of A an arrow $\alpha_a : \Phi(a) \to \Psi(a)$ of B in such a way that the well–known conditions defining a natural transformation of functors is fulfilled only up to a 2–cells of B again satisfying certain coherence conditions. A set of smooth data $\{h_i, J_{ij}\}$ satisfying (3.7.22) precisely defines a smooth pseudonatural isomorphism of the pseudofunctors (g, W), (g', W') mapping the each 0–cell (m, i) to the 1–cell $h_i(m)$ intertwining the 1–cells $g_{ji}(m)$, $g'{}_{ji}(m)$ corresponding to the (m, i, j) only up to the 2–cells $J_{ij} : g_{ij} \circ h_j \Rightarrow h_i \circ g'{}_{ij}$.

In this way, a bijection is established between the isomorphism classes pf smooth principal V–2–bundles P and the pseudonatural isomorphism classes of pseudofunctors $\check{C}_2(U) \to BV$. The theory of such principal 2–bundle classes is therefore completely encoded in the pseudofunctor category $[\check{C}_2(U), BV]$.

3.8 2–term L_∞ algebra gauge theory, global aspects

Now, we have all the elements necessary for the analysis of the global aspects of semistrict higher gauge theory.

Let M be a smooth d–fold. Though M is not necessarily diffeomorphic to $\mathbb{R}^d$, it admits an open covering $\{U_i\}$ such that the all sets U_i as well as all their non empty finite intersections are.

To understand fully how things work in higher gauge theory, we begin again with considering what happens in an ordinary gauge theory with structure Lie algebra $\mathfrak{g}$. A generic field $\mathcal{F}$ on M is not in general a vector valued function globally defined on M, but instead is given as a collection $\{\mathcal{F}_i\}$, where $\mathcal{F}_i$ is a vector valued function defined locally on U_i. $\mathcal{F}_i$ can be viewed as the representation of $\mathcal{F}$ with respect to a local vector frame on U_i. $\mathcal{F}_i$, $\mathcal{F}_j$ are thus related by a frame change on every U_{ij}. The $\mathcal{F}_i$ are typically $\mathfrak{g}$ valued fields and are so

acted upon by the gauge transformation group $\mathrm{Gau}(U_i, \mathfrak{g})$ (cf. subsect. 3.4). It is natural to require that the frame change occurring on each U_{ij} is given by a gauge transformation $g_{ij} \in \mathrm{Gau}(U_{ij}, \mathfrak{g})$, so that we have

$$\mathcal{F}_i = {}^{g_{ij}}\mathcal{F}_j, \qquad \text{on } U_{ij}. \tag{3.8.1}$$

The gluing data g_{ij} are defined up to a certain equivalence relation amd must satisfy a coherence condition analogous to those of principal bundle theory. To study the associated class of topological and geometrical structures, we introduce a groupoid $\check{\mathcal{P}}(U, \mathfrak{g})$ defined analogously to the groupoid $\check{P}(U, G)$ of subsect. 3.7 by replacing the mapping group $\mathrm{Map}(\cdot, G)$ by the gauge transformation group $\mathrm{Gau}(\cdot, \mathfrak{g})$ throughout (cf. subsect. 3.3). $\check{\mathcal{P}}(U, \mathfrak{g})$ can thus be described in the following terms.

1. A 0–cell of $\check{\mathcal{P}}(U, \mathfrak{g})$ is collection $g = \{g_{ij}\}$ of gauge transformations $g_{ij} \in \mathrm{Gau}(U_{ij}, \mathfrak{g})$ satisfying the condition

$$g_{ij} \diamond g_{jk} = g_{ik}, \qquad \text{on } U_{ijk}. \tag{3.8.2}$$

2. For any two 0–cells g, g' of $\check{\mathcal{P}}(U, \mathfrak{g})$, a 1–cell $g \to g'$ is a collection $\{h_i\}$ of gauge transformations $h_i \in \mathrm{Gau}(U_i, \mathfrak{g})$ such that

$$g_{ij} \diamond h_j = h_i \diamond g'_{ij}, \qquad \text{on } U_{ij}. \tag{3.8.3}$$

Further, the groupoid operations are defined formally in the same way as those of $\check{P}(U, G)$.

Concretely, a 0–cell g is equivalent to a collection of data $\{g_{ij}, \sigma_{ij}\}$ with $g_{ij} \in \mathrm{Map}(U_{ij}, \mathrm{Aut}(\mathfrak{g}))$ and σ_{ij} a connection on U_{ij} such that

$$d\sigma_{ij} + \frac{1}{2}[\sigma_{ij}, \sigma_{ij}] = 0, \tag{3.8.4a}$$

$$g_{ij}{}^{-1} dg_{ij}(x) - [\sigma_{ij}, x] = 0 \tag{3.8.4b}$$

satisfying the coherence conditions

$$g_{ij} g_{jk} = g_{ik}, \tag{3.8.5a}$$

$$\sigma_{ik} - \sigma_{jk} - {g_{jk}}^{-1}(\sigma_{ij}) = 0. \tag{3.8.5b}$$

For any two 0–cell g, g' of $\check{\mathcal{P}}(U, \mathfrak{g})$, a 1–cell $h : g \to g'$ is equivalent to a collection of data $\{h_i, \pi_i\}$ with $h_i \in \mathrm{Map}(U_i, \mathrm{Aut}(\mathfrak{g}))$ and π_i a connection on U_i with

$$d\pi_i + \frac{1}{2}[\pi_i, \pi_i] = 0, \tag{3.8.6a}$$

$$h_i^{-1} dh_i(x) - [\pi_i, x] = 0 \tag{3.8.6b}$$

satisfying the coherence conditions

$$g_{ij} h_j = h_i {g'}_{ij}, \tag{3.8.7a}$$

$$\sigma'_{ij} - {h_j}^{-1}(\sigma_{ij}) - \pi_j + {g'_{ij}}^{-1}(\pi_i) = 0. \tag{3.8.7b}$$

By construction, the above topological set up is formally analogous to that of principal bundle theory. Yet it is not possible to frame it in principal bundle theoretic terms. Any attempt at casting the groupoid $\check{\mathcal{P}}(U, \mathfrak{g})$ as a groupoid of the form $\check{P}(U, \widehat{G})$ for some Lie group $\widehat{G}$ and interpret 1–cell isomorphism classes of 0–cells of $\check{\mathcal{P}}(U, \mathfrak{g})$ as ones of $\check{P}(U, \widehat{G})$, hence as isomorphisms classes of principal $\widehat{G}$–bundles, will fail because of the differential conditions (3.8.4), (3.8.6) obeyed by the 1–form data. We note however that a 1–cell isomorphism class of 0–cells in $\check{\mathcal{P}}(U, \mathfrak{g})$ yields a 1–cell isomorphism class of 0–cells in $\check{P}(U, \mathrm{Aut}(\mathfrak{g}))$ obtained by the forgetful map that keeps the data g_{ij} and h_i but drops the data σ_{ij} and π_i, thus an isomorphism class of principal $\mathrm{Aut}(\mathfrak{g})$–bundles. Thus, a principal $\mathrm{Aut}(\mathfrak{g})$–bundle is part of the structure defined by the 0–cell, but it does exhaust it. Similar remarks holds for the counterpart of principal bundle automorphisms.

It is important to emphasize the points where the above gauge theoretic framework differs from the standard one. Consider a conventional gauge theory with

structure group G. The topological background of the theory is then a principal G–bundle P codified in a 1–cell isomorphism class of 0–cells $\gamma = \{\gamma_{ij}\}$ of $\check{P}(U,G)$. Since in gauge theory all fields are in the adjoint of G, the effective structure group is the adjoint group $\mathrm{Ad}\,G = G/Z(G)$ rather than G. Can we, then, replace G by $\mathrm{Ad}\,G$? The crucial point is whether from the knowledge of the data $g_{ij} = \mathrm{Ad}\,\gamma_{ij}$ it is possible to reconstruct the data $\sigma_{ij} = {\gamma_{ij}}^{-1} d\gamma_{ij}$ which control the global matching of local connections. It depends on the structure group G. If G is semisimple, e. g. $G = \mathrm{SU}(n)$, it can indeed be done. If G is not semisimple, e. g. $G = \mathrm{U}(1)$, then it is no longer possible. However, we can still work with $\mathrm{Ad}\,G$ rather than G, if we give up the condition that the σ_{ij} be determined by the g_{ij} and regard the former as data (partially) independent from the latter, switching from $\check{P}(U,G)$ to $\check{\mathcal{P}}(U,\mathfrak{g})$ and controlling the global definedness of the fields using 1–cell isomorphism class of 0–cells $g = \{g_{ij}, \sigma_{ij}\}$. It is clear that this can only work perturbatively, as important non perturbative effects are attached to the center $Z(G)$ of G. As $\mathrm{Ad}\,G \simeq \mathrm{Inn}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$, however, by working with $\check{\mathcal{P}}(U,\mathfrak{g})$ we are generalizing gauge theory including the case where the data g_{ij} take values in the non inner automorphisms of $\mathfrak{g}$.

Let us now assume that the global properties of the fields of a gauge theory with structure Lie algebra $\mathfrak{g}$ of the generalized sort described in the previous paragraph are defined by a 1–cell isomorphism class of 0–cells $g = \{g_{ij}, \sigma_{ij}\}$ of $\check{\mathcal{P}}(U,\mathfrak{g})$.

A connection ω on M is a collection $\{\omega_i\}$ of connections ω_i on the sets U_i satisfying the matching relation (3.8.1) with $\mathcal{F} = \omega$, the right hand side being given by (3.4.1). The curvature f of ω on M is the collection $\{f_i\}$ of the curvatures f_i of the ω_i and satisfies (3.8.1) with $\mathcal{F} = f$ and the right hand side given by (3.4.2).

A field ϕ on M is a collection $\{\phi_i\}$ of fields ϕ_i of the same type on the sets U_i satisfying matching relations of the form (3.8.1) with $\mathcal{F} = \phi$, the right hand

side being given by (3.4.3). The covariant derivative field $D\phi$ of ϕ on M is the collection $\{D\phi_i\}$ of the covariant derivative fields $D\phi_i$ of the ϕ_i and satisfies also (3.8.1) with $\mathcal{F} = D\phi$ and the right hand side given by (3.4.4).

Similarly, a dual field υ on M is a collection $\{\upsilon_i\}$ of dual fields υ_i of the same type on the sets U_i such that (3.8.1) holds with $\mathcal{F} = \upsilon$ and the right hand side given by (3.4.5). The covariant derivative dual field $D\upsilon$ on M is the collection $\{D\upsilon_i\}$ of the covariant derivative dual fields $D\upsilon_i$ of the υ_i and satisfies matching relations of the form (3.8.1) with $\mathcal{F} = D\upsilon$ and the right hand side given by (3.4.6).

In a 2–term L_∞ algebra gauge theory with structure algebra $\mathfrak{v}$, things proceed much in the same way. A generic field $\mathcal{F}$ on M is again a collection $\{\mathcal{F}_i\}$ of local fields $\mathcal{F}_i$. Here, the $\mathcal{F}_i$ are fields of one of the types considered in subsect. 3.2 and so are acted upon by the 1–gauge transformation group $\mathrm{Gau}_1(U_i, \mathfrak{v})$ (cf. subsect. 3.4). It is natural to require that the relationship of the local fields $\mathcal{F}_i$, $\mathcal{F}_j$ on U_{ij} is given by a 1–gauge transformation $g_{ij} \in \mathrm{Gau}_1(U_{ij}, \mathfrak{v})$,

$$\mathcal{F}_i = {}^{g_{ij}}\mathcal{F}_j, \qquad \text{on } U_{ij}. \tag{3.8.8}$$

in analogy to (3.8.1). Again, as in principal bundle theory, the gluing data g_{ij} are defined up to a certain equivalence relation and must satisfy a compatibility condition. This latter, however, cannot be expressed in terms of the g_{ij} only but requires the introduction of further gluing data not entering (3.8.8), 2–gauge transformations $W_{ijk} \in \mathrm{Gau}_2(U_{ijk}, \mathfrak{v})$, satisfying a coherence condition. In analogy with ordinary gauge theory, to study the associated class of topological and geometrical structures, we introduce a strict 2–groupoid $\check{\mathcal{P}}_2(U, \mathfrak{v})$ defined analogously to the groupoid $\check{P}_2(U, V)$ studied in subsect. 3.7 by replacing the mapping strict 2–group $\mathrm{Map}(\cdot, V)$ by the 2–term L_∞ algebra gauge transformation 2–group $\mathrm{Gau}(\cdot, \mathfrak{v})$ throughout (cf. subsect. 3.3). $\check{\mathcal{P}}_2(U, \mathfrak{v})$ can be described in the following terms.

1. A 0–cell of $\check{\mathcal{P}}_2(U,\mathfrak{v})$ is a collection $(g,W)=\{g_{ij},W_{ijk}\}$ of 1–gauge transformations $g_{ij}\in\mathrm{Gau}_1(U_{ij},\mathfrak{v})$ and 2–gauge transformations $W_{ijk}\in\mathrm{Gau}_2(U_{ijk},\mathfrak{v})$ such that one has

$$W_{ijk}:g_{ik}\Rightarrow g_{ij}\diamond g_{jk},\qquad \text{on } U_{ijk}, \tag{3.8.9a}$$

where $\diamond$ denotes the composition of 1–cells in $\mathrm{Gau}_1(\cdot,\mathfrak{v})$, and satisfying the coherence condition

$$(I_{g_{ij}}\diamond W_{jkl})\bullet W_{ijl}=(W_{ijk}\diamond I_{g_{kl}})\bullet W_{ikl}.\qquad \text{on } U_{ijkl}, \tag{3.8.9b}$$

where $\diamond$ and $\bullet$ denote the horizontal and vertical composition of 2–cells in $\mathrm{Gau}_2(\cdot,\mathfrak{v})$, respectively (cf. subsect. 3.3).

2. For any two 0–cells (g,W), (g',W'), a 1–cell $(g,W)\to(g',W')$ of $\check{\mathcal{P}}_2(U,\mathfrak{v})$ is a collection $(h,J)=\{h_i,J_{ij}\}$ of 1–gauge transformations $h_i\in\mathrm{Gau}_1(U_i,\mathfrak{v})$ and 2–gauge transformations $J_{ij}\in\mathrm{Gau}_2(U_{ij},\mathfrak{v})$ such that

$$J_{ij}:g_{ij}\diamond h_j\Rightarrow h_i\diamond {g'}_{ij},\qquad \text{on } U_{ij}, \tag{3.8.10a}$$

satisfying the coherence condition

$$(J_{ij}\diamond I_{{g'}_{jk}})\bullet(I_{g_{ij}}\diamond J_{jk})\bullet(W_{ijk}\diamond I_{h_k})=(I_{h_i}\diamond {W'}_{ijk})\bullet J_{ik},\quad \text{on } U_{ijk}. \tag{3.8.10b}$$

3. For any two 1–cells $(h,J),(h',J'):(g,W)\to(g',W')$, a 2–cell $(h,J)\Rightarrow(h',J')$ of $\check{\mathcal{P}}_2(U,\mathfrak{v})$ is a collection $K=\{K_i\}$ of 2–gauge transformations $K_i\in\mathrm{Gau}_2(U_i,\,\mathfrak{v})$ satisfying

$$K_i:h_i\Rightarrow {h'}_i,\qquad \text{on } U_i, \tag{3.8.11a}$$

and the coherence condition

$${J'}_{ij}\bullet(I_{g_{ij}}\diamond K_j)=(K_i\diamond I_{{g'}_{ij}})\bullet J_{ij},\qquad \text{on } U_{ij}. \tag{3.8.11b}$$

The 2–groupoid operations are defined formally in the same way as in $\check{P}_2(U,V)$.

Concretely, a 0–cell (g,W) is equivalent to a collection of data $\{g_{ij},\sigma_{ij},\Sigma_{ij},\tau_{ij},W_{ijk},A_{ijk}\}$ with $g_{ij}\in\mathrm{Map}(U_{ij},\mathrm{Aut}_1(\mathfrak{v}))$, $(\sigma_{ij},\Sigma_{ij})$ a connection doublet on U_{ij}, $\tau_{ij}\in\Omega^1(U_{ij},\mathrm{Hom}(\hat{\mathfrak{v}}_0,\hat{\mathfrak{v}}_1))$, $W_{ijk}\in\mathrm{Map}(U_{ijk},\mathrm{Aut}_2(\mathfrak{v}))$ and $A_{ijk}\in\Omega^1(U_{ijk},\hat{\mathfrak{v}}_1)$ satisfying the relations

$$d\sigma_{ij}+\frac{1}{2}[\sigma_{ij},\sigma_{ij}]-\partial\Sigma_{ij}=0, \tag{3.8.12a}$$

$$d\Sigma_{ij}+[\sigma_{ij},\Sigma_{ij}]-\frac{1}{6}[\sigma_{ij},\sigma_{ij},\sigma_{ij}]=0, \tag{3.8.12b}$$

$$\begin{aligned}d\tau_{ij}(x)+[\sigma_{ij},\tau_{ij}(x)]-[x,\Sigma_{ij}]+\frac{1}{2}[\sigma_{ij},\sigma_{ij},x]&\\ +\tau_{ij}([\sigma_{ij},x]+\partial\tau_{ij}(x))=0,&\end{aligned} \tag{3.8.12c}$$

$${g_{ij0}}^{-1}dg_{ij0}(x)-[\sigma_{ij},x]-\partial\tau_{ij}(x)=0, \tag{3.8.12d}$$

$${g_{ij1}}^{-1}dg_{ij1}(X)-[\sigma_{ij},X]-\tau_{ij}(\partial X)=0, \tag{3.8.12e}$$

$$\begin{aligned}{g_{ij1}}^{-1}(dg_{ij2}(x,y)-g_{ij2}({g_{ij0}}^{-1}dg_{ij0}(x),y)-g_{ij2}(x,{g_{ij0}}^{-1}dg_{ij0}(y)))&\\ -[\sigma_{ij},x,y]-\tau_{ij}([x,y])+[x,\tau_{ij}(y)]-[y,\tau_{ij}(x)]=0,&\end{aligned} \tag{3.8.12f}$$

$$W_{ijk}:g_{ik}\Rightarrow g_{ij}\circ g_{jk}, \tag{3.8.12g}$$

$$\sigma_{jk}-\sigma_{ik}+{g_{jk0}}^{-1}(\sigma_{ij})+\partial A_{ijk}=0, \tag{3.8.12h}$$

$$\begin{aligned}\Sigma_{jk}-\Sigma_{ik}+{g_{jk1}}^{-1}(\Sigma_{ij})+\frac{1}{2}{g_{jk1}}^{-1}g_{jk2}({g_{jk0}}^{-1}(\sigma_{ij}),{g_{jk0}}^{-1}(\sigma_{ij}))&\\ -\tau_{jk}({g_{jk0}}^{-1}(\sigma_{ij}))+dA_{ijk}+[\sigma_{ik},A_{ijk}]-\frac{1}{2}[\partial A_{ijk},A_{ijk}]=0,&\end{aligned} \tag{3.8.12i}$$

$$\begin{aligned}\tau_{jk}(x)-\tau_{ik}(x)+{g_{jk1}}^{-1}(\tau_{ij}(g_{jk0}(x)))-{g_{jk1}}^{-1}g_{jk2}({g_{jk0}}^{-1}(\sigma_{ij}),x)&\\ +[x,A_{ijk}]+(g_{ij1}g_{jk1})^{-1}\big(dW_{ijk}(x)-W_{ijk}([\sigma_{ik},x]+\partial\tau_{ik}(x))\big)=0.&\end{aligned} \tag{3.8.12j}$$

and satisfying the coherence conditions

$$(1_{g_{ij}}\circ W_{jkl})\cdot W_{ijl}=(W_{ijk}\circ 1_{g_{kl}})\cdot W_{ikl}, \tag{3.8.13a}$$

$$A_{jkl}-A_{ikl}+A_{ijl}-{g_{kl1}}^{-1}(A_{ijk})-{g_{jl1}}^{-1}W_{jkl}(g_{jk0}g_{kl0})^{-1}(\sigma_{ij})=0. \tag{3.8.13b}$$

For any two 0–cells (g,W), (g',W'), a 1–cell $(g,W)\to(g',W')$ is equivalent to

a collection of data $\{h_i, \pi_i, \Pi_i, \rho_i, J_{ij}, D_{ij}\}$ with $h_i \in \mathrm{Map}(U_i, \mathrm{Aut}_1(\mathfrak{v}))$, (π_i, Π_i) a connection doublet on U_i, $\rho_i \in \Omega^1(U_i, \mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1))$, $J_{ij} \in \mathrm{Map}(U_{ij}, \mathrm{Aut}_2(\mathfrak{v}))$ and $D_{ij} \in \Omega^1(U_{ij}, \hat{\mathfrak{v}}_1)$ satisfying the relations

$$d\pi_i + \frac{1}{2}[\pi_i, \pi_i] - \partial \Pi_i = 0, \tag{3.8.14a}$$

$$d\Pi_i + [\pi_i, \Pi_i] - \frac{1}{6}[\pi_i, \pi_i, \pi_i] = 0, \tag{3.8.14b}$$

$$\begin{aligned} d\rho_i(x) + [\pi_i, \rho_i(x)] - [x, \Pi_i] + \frac{1}{2}[\pi_i, \pi_i, x] \\ + \rho_i([\pi_i, x] + \partial\rho_i(x)) = 0, \end{aligned} \tag{3.8.14c}$$

$$h_{i0}{}^{-1} dh_{i0}(x) - [\pi_i, x] - \partial\rho_i(x) = 0, \tag{3.8.14d}$$

$$h_{i1}{}^{-1} dh_{i1}(X) - [\pi_i, X] - \rho_i(\partial X) = 0, \tag{3.8.14e}$$

$$\begin{aligned} h_{i1}{}^{-1}(dh_{i2}(x, y) - h_{i2}(h_{i0}{}^{-1} dh_{i0}(x), y) - h_{i2}(x, h_{i0}{}^{-1} dh_{i0}(y))) \\ - [\pi_i, x, y] - \rho_i([x, y]) + [x, \rho_i(y)] - [y, \rho_i(x)] = 0, \end{aligned} \tag{3.8.14f}$$

$$J_{ij} : g_{ij} \circ h_j \Rightarrow h_i \circ g'_{ij}, \tag{3.8.14g}$$

$$\sigma'_{ij} - h_{j0}{}^{-1}(\sigma_{ij}) - \pi_j + g'_{ij0}{}^{-1}(\pi_i) + \partial D_{ij} = 0, \tag{3.8.14h}$$

$$\begin{aligned} &\Sigma'_{ij} - h_{j1}{}^{-1}(\Sigma_{ij}) - \frac{1}{2} h_{j1}{}^{-1} h_{j2}(h_{j0}{}^{-1}(\sigma_{ij}), h_{j0}{}^{-1}(\sigma_{ij})) \\ &+ \rho_j(h_{j0}{}^{-1}(\sigma_{ij})) - \Pi_j + g'_{ij1}{}^{-1}(\Pi_i) + \frac{1}{2} g'_{ij1}{}^{-1} g'_{ij2}(g'_{ij0}{}^{-1}(\pi_i), g'_{ij0}{}^{-1}(\pi_i)) \\ &- \tau'_{ij}(g'_{ij0}{}^{-1}(\pi_i)) + dD_{ij} + [\sigma'_{ij} + g'_{ij0}{}^{-1}(\pi_i), D_{ij}] + \frac{1}{2}[\partial D_{ij}, D_{ij}] = 0, \end{aligned} \tag{3.8.14i}$$

$$\begin{aligned} &\tau'_{ij}(x) - h_{j1}{}^{-1}(\tau_{ij}(h_{j0}(x))) + h_{j1}{}^{-1} h_{j2}(h_{j0}{}^{-1}(\sigma_{ij}), x) \\ &- \rho_j(x) + g'_{ij1}{}^{-1}(\rho_i(g'_{ij0}(x))) - g'_{ij1}{}^{-1} g'_{ij2}(g'_{ij0}{}^{-1}(\pi_i), x) \\ &+ [x, D_{ij}] + (g_{ij1} h_{j1})^{-1}\big(dJ_{ij}(x) - J_{ij}\big([\sigma'_{ij} + g'_{ij0}{}^{-1}(\pi_i), x] \\ &+ \partial(\tau'_{ij}(x) + g'_{ij1}{}^{-1}(\rho_i(g'_{ij0}(x))) - g'_{ij1}{}^{-1} g'_{ij2}(g'_{ij0}{}^{-1}(\pi_i), x))\big)\big) = 0 \end{aligned} \tag{3.8.14j}$$

and satisfying the coherence conditions

$$(J_{ij} \circ 1_{g'_{jk}}) \cdot (1_{g_{ij}} \circ J_{jk}) \cdot (W_{ijk} \circ 1_{h_k}) = (1_{h_i} \circ W'_{ijk}) \cdot J_{ik}, \tag{3.8.15a}$$

$$D_{jk} - D_{ik} + {g'_{jk1}}^{-1}(D_{ij}) - (g_{jk1}h_{k1})^{-1}J_{jk}(h_{j0}{g'}_{jk0})^{-1}(\sigma_{ij}) \tag{3.8.15b}$$
$$+ {g'_{ik1}}^{-1}{W'}_{ijk}({g'}_{ij0}{g'}_{jk0})^{-1}(\pi_i) - {A'}_{ijk} + {h_{k1}}^{-1}(A_{ijk}) = 0$$

For any two 1–cells $(h, J), (h', J') : (g, W) \to (g', W')$, a 2–cell $(h, J) \Rightarrow (h', J')$ is equivalent to a collection of data $\{K_i, C_i\}$ with $K_i \in \mathrm{Map}(U_i, \mathrm{Aut}_2(\mathfrak{v}))$ and $C_i \in \Omega^1(U_i, \hat{\mathfrak{v}}_1)$ satisfying the relations

$$K_i : h_i \Rightarrow {h'}_i \tag{3.8.16a}$$

$$\pi_i - {\pi'}_i = \partial C_i, \tag{3.8.16b}$$

$$\mathit{\Pi}_i - {\mathit{\Pi}'}_i = dC_i + [{\pi'}_i, C_i] + \frac{1}{2}[\partial C_i, C_i], \tag{3.8.16c}$$

$$\rho_i(x) - {\rho'}_i(x) = [x, C_i] + {h_{i1}}^{-1}(dK_i(x) - K_i([{\pi'}_i, x] + \partial{\rho'}_i(x))). \tag{3.8.16d}$$

and satisfying the coherence condition

$$D'_{ij} - D_{ij} + C_j - {g'_{ij1}}^{-1}(C_i) - {h_{j1}}^{-1}K_j{h'_{j0}}^{-1}(\sigma_{ij}) = 0. \tag{3.8.17}$$

By construction, the above topological set up is formally analogous to that of principal 2–bundle theory of subsect. 3.7, but, similarly to ordinary gauge theory, it is not possible to frame it in principal 2–bundle theoretic terms. The 2–groupoid $\check{\mathcal{P}}_2(U, \mathfrak{v})$ cannot be cast as a 2–groupoid of the form $\check{P}_2(U, \widehat{V})$ for some strict Lie 2–group $\widehat{V}$, 1–cell isomorphism classes of 0–cells of $\check{\mathcal{P}}_2(U, \mathfrak{v})$ cannot be interpreted as ones of $\check{P}_2(U, \widehat{V})$, hence as isomorphism classes of principal $\widehat{V}$–2–bundles, because of the differential conditions (3.8.12), (3.8.14) obeyed by the 1–and 2 form data. Similarly to ordinary gauge theory again, a 1–cell isomorphism class of 0–cells in $\check{\mathcal{P}}_2(U, \mathfrak{v})$ yields a 1–cell isomorphism class of 0–cells in $\check{P}_2(U, \mathrm{Aut}(\mathfrak{v}))$ obtained by the forgetful map that keeps the data g_{ij}, W_{ijk} and h_i, J_{ij} but drops the data σ_{ij}, $\mathit{\Sigma}_{ij}$, τ_{ij}, A_{ijk} and π_i, $\mathit{\Pi}_i$, ρ_i, D_{ij}, thus an isomorphism class of principal $\mathrm{Aut}(\mathfrak{v})$–2–bundles. In this way, a principal $\mathrm{Aut}(\mathfrak{v})$–2–bundle is part of the set up without exhausting it. Similar remarks hold for the counterpart of principal 2–bundle automorphisms and automorphism for automorphisms.

It is important to relate the above gauge theoretic framework with others which have appeared previously in the literature, in particular [32, 33] (see also [34]). Consider a higher gauge theory with a strict structure 2–group V. The topological background of the theory is then a principal V–2–bundle P codified in a 1–cell isomorphism class of 0–cells $(\gamma, \varTheta) = \{\gamma_{ij}, \varTheta_{ijk}\}$ of $\check{P}_2(U, G, H)$, where (G, H) is the Lie crossed module corresponding to V (cf. subsect. 3.7). Unlike standard gauge theory, the data γ_{ij}, $\varTheta_{ijk}$ are not sufficient by themselves to control the global matching of local connections. Further independent data $\chi_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$ are required (cf. subsect. 3.6). To frame all these data in a coherent whole and to study the associated class of topological and geometrical structures, we introduce a strict 2–groupoid $\check{\mathcal{P}}_2(U, G, H)$ defined analogously to the groupoid $\check{P}_2(U, G, H)$ by replacing the mapping strict 2–group $\mathrm{Map}(\cdot, G, H)$ by the crossed module gauge transformation strict 2–group $\mathrm{Gau}(\cdot, G, H)$ studied in subsect. 3.6 throughout. $\check{\mathcal{P}}_2(U, G, H)$ can be described as follows.

1. A 0–cell of $\check{\mathcal{P}}_2(U, G, H)$ is a collection $(\gamma, \varTheta) = \{\gamma_{ij}, \varTheta_{ijk}\}$ of 1–gauge transformations $\gamma_{ij} \in \mathrm{Gau}_1(U_{ij}, G, H)$ and 2–gauge transformations $\varTheta_{ijk} \in \mathrm{Gau}_2(U_{ijk}, G, H)$ such that

$$\varTheta_{ijk} : \gamma_{ik} \Rightarrow \gamma_{ij} \diamond \gamma_{jk}, \qquad \text{on } U_{ijk}, \tag{3.8.18a}$$

   where $\diamond$ denotes the composition of 1–cells in $\mathrm{Gau}_1(\cdot, G, H)$, and satisfying the coherence condition

$$(I_{\gamma_{ij}} \diamond \varTheta_{jkl}) \bullet \varTheta_{ijl} = (\varTheta_{ijk} \diamond I_{\gamma_{kl}}) \bullet \varTheta_{ikl}. \qquad \text{on } U_{ijkl}, \tag{3.8.18b}$$

   where $\diamond$ and $\bullet$ denote the horizontal and vertical composition of 2–cells in $\mathrm{Gau}_2(\cdot, G, H)$, respectively (cf. subsect. 3.6).

2. For any two 0–cells $(\gamma, \varTheta)$, $(\gamma', \varTheta')$, a 1–cell $(\gamma, \varTheta) \to (\gamma', \varTheta')$ of $\check{\mathcal{P}}_2(U, G, H)$ is a collection $(\eta, \varUpsilon) = \{\eta_i, \varUpsilon_{ij}\}$ of 1–gauge transformations $\eta_i \in \mathrm{Gau}_1(U_i,$

$G, H)$ and 2–gauge transformations $\Upsilon_{ij} \in \mathrm{Gau}_2(U_{ij}, G, H)$ such that

$$\Upsilon_{ij} : \gamma_{ij} \diamond \eta_j \Rightarrow \eta_i \diamond {\gamma'}_{ij}, \qquad \text{on } U_{ij}, \tag{3.8.19a}$$

satisfying the coherence condition

$$(\Upsilon_{ij} \diamond I_{{\gamma'}_{jk}}) \bullet (I_{\gamma_{ij}} \diamond \Upsilon_{jk}) \bullet (\Theta_{ijk} \diamond I_{\eta_k}) = (I_{\eta_i} \diamond {\Theta'}_{ijk}) \bullet \Upsilon_{ik}, \quad \text{on } U_{ijk}. \tag{3.8.19b}$$

3. For any two 1–cells $(\eta, \Upsilon), (\eta', \Upsilon') : (\gamma, \Theta) \to (\gamma', \Theta')$, a 2–cell $(\eta, \Upsilon) \Rightarrow (\eta', \Upsilon')$ of $\check{\mathcal{P}}_2(U, G, H)$ is a collection $\Lambda = \{\Lambda_i\}$ of 2–gauge transformations $\Lambda_i \in \mathrm{Gau}_2(U_i,\, G, H)$ satisfying

$$\Lambda_i : \eta_i \Rightarrow {\eta'}_i, \qquad \text{on } U_i, \tag{3.8.20a}$$

and the coherence condition

$$\Upsilon'_{ij} \bullet (I_{\gamma_{ij}} \diamond \Lambda_j) = (\Lambda_i \diamond I_{{\gamma'}_{ij}}) \bullet \Upsilon_{ij}, \qquad \text{on } U_{ij}. \tag{3.8.20b}$$

The 2–groupoid operations are defined formally in the same way as $\check{P}_2(U, G, H)$.

Concretely, a 0–cell (γ, Θ) is equivalent to a collection of data $\{\gamma_{ij}, \chi_{ij}, \Theta_{ijk}, B_{ijk}\}$ with $\gamma_{ij} \in \mathrm{Map}(U_{ij}, G)$ $\chi_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$, $\Theta_{ijk} \in \mathrm{Map}(U_{ijk}, H)$, $B_{ijk} \in \Omega^1(U_{ijk}, \mathfrak{h})$ satisfying the relations

$$\gamma_{ij}\gamma_{jk} = t(\Theta_{ijk})\gamma_{ik}, \tag{3.8.21a}$$

$$\dot{m}(\gamma_{ij})(\chi_{jk}) - \chi_{ik} + \chi_{ij} + B_{ijk} = 0 \tag{3.8.21b}$$

and the coherence conditions

$$m(\gamma_{ij})(\Theta_{jkl})\Theta_{ijl} = \Theta_{ijk}\Theta_{ikl}, \tag{3.8.22a}$$

$$\dot{m}(\gamma_{ij})(B_{jkl}) - B_{ikl} + B_{ijl} - B_{ijk} - Q(\gamma_{ik}\dot{t}(\chi_{kl}){\gamma_{ik}}^{-1}, \Theta_{ijk}) = 0. \tag{3.8.22b}$$

For any two 0–cells (γ, Θ), (γ', Θ'), a 1–cell $(\eta, \Upsilon) : (\gamma, \Theta) \to (\gamma', \Theta')$ is the

same as a collection of data $\{\eta_i, \lambda_i, \Upsilon_{ij}, E_{ij}\}$ with $\eta_i \in \mathrm{Map}(U_i, G)$, $\lambda_i \in \Omega^1(U_i, \mathfrak{h})$, $\Upsilon_{ij} \in \mathrm{Map}(U_{ij}, H)$, $E_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$ such that

$$\eta_i {\gamma'}_{ij} = t(\Upsilon_{ij})\gamma_{ij}\eta_j, \tag{3.8.23a}$$

$$\dot{m}(\eta_i)({\chi'}_{ij}) - \chi_{ij} - \dot{m}(\gamma_{ij})(\lambda_j) + \lambda_i + E_{ij} = 0, \tag{3.8.23b}$$

and the coherence conditions

$$\Upsilon_{ij} m(\gamma_{ij})(\Upsilon_{jk})\Theta_{ijk} = m(\eta_i)({\Theta'}_{ijk})\Upsilon_{ik}, \tag{3.8.24a}$$

$$\begin{aligned}\dot{m}(\eta_i)({B'}_{ijk}) - B_{ijk} - \dot{m}(\gamma_{ij})(E_{jk}) + E_{ik} - E_{ij}& \\ - Q(\gamma_{ij}\eta_j\dot{t}(\chi_{kl}){\eta_j}^{-1}{\gamma_{ij}}^{-1}, \Upsilon_{ij}) - Q(\gamma_{ik}\dot{t}(\lambda_k){\gamma_{ik}}^{-1}, \Theta_{ijk}) &= 0.\end{aligned} \tag{3.8.24b}$$

Finally, for any two 1–cells $(\eta, \Upsilon), (\eta', \Upsilon') : (\gamma, \Theta) \to (\gamma', \Theta')$, a 2–cell $\Lambda : (\eta, \Upsilon) \Rightarrow (\eta', \Upsilon')$ is equivalent to a collection of data $\{\Lambda_i, L_i\}$ with $\Lambda_i \in \mathrm{Map}(U_i, H)$, $L_i \in \Omega^1(U_i, \mathfrak{h})$ satisfying the relations

$${\eta'}_i = t(\Lambda_i)\eta_i, \tag{3.8.25a}$$

$$\lambda_i - {\lambda'}_i = L_i, \tag{3.8.25b}$$

and the coherence conditions

$${\Upsilon'}_{ij} m(\gamma_{ij})(\Lambda_j) = \Lambda_i \Upsilon_{ij}, \tag{3.8.26a}$$

$${E'}_{ij} - E_{ij} - \dot{m}({\gamma'}_{ij})(L_j) + L_i - Q(\eta_i\dot{t}({\chi'}_{ij}){\eta_i}^{-1}, \Lambda_i) = 0. \tag{3.8.26b}$$

Here, unlike the cases considered before, there are no differential conditions which the 1–form data must obey. It should then be possible to cast 2–groupoid $\check{\mathcal{P}}_2(U, G, H)$ as a 2–groupoid of the form $\check{P}_2(U, \widehat{G}, \widehat{H})$ for some Lie crossed module $(\widehat{G}, \widehat{H})$ and 1–cell isomorphism classes of 0–cells of $\check{\mathcal{P}}_2(U, G, H)$ may be interpreted as ones of $\check{P}_2(U, \widehat{G}, \widehat{H})$, hence as isomorphism classes of principal $(\widehat{G}, \widehat{H})$–2–bundles. We shall not attempt to describe $(\widehat{G}, \widehat{H})$. We shall limit ourselves to note that a 1–cell isomorphism class of 0–cells in $\check{\mathcal{P}}_2(U, G, H)$ yields a 1–cell iso-

morphism class of 0–cells in $\check{P}_2(U,G,H)$ obtained by the forgetful map that keeps the data γ_{ij}, Θ_{ijk} and η_i, Υ_{ij} dropping the data χ_{ij}, B_{ijk}, λ_i, E_{ij}, the isomorphism class of principal (G,H)–2–bundles which we started with.

In subsect. 3.6, we have shown that there is a natural strict 2–group 1–morphism from the crossed module gauge transformation 2–group $\mathrm{Gau}(M,G,H)$ to the gauge transformation 2–group $\mathrm{Gau}(M,\mathfrak{v})$, where $\mathfrak{v}$ is the strict 2–term L_∞ algebra corresponding to the differential Lie crossed module $(\mathfrak{g},\mathfrak{h})$. Now, the operations of the 2–groupoids $\check{\mathcal{P}}_2(U,G,H)$ and $\check{\mathcal{P}}_2(U,\mathfrak{v})$ are defined completely in terms of those of the 2–groups $\mathrm{Gau}(M,G,H)$ and $\mathrm{Gau}(M,\mathfrak{v})$, respectively. A natural strict 2–groupoid 1–morphism from $\check{\mathcal{P}}_2(U,G,H)$ to $\check{\mathcal{P}}_2(U,\mathfrak{v})$ is thus induced, furnishing us with a dictionary translating the formulation of higher gauge of refs. [32,33] extended in the way we have indicated into the one worked out in this paper. This parallels what happens in ordinary gauge theory, though in a rather non trivial way. As in ordinary gauge theory, working with $\check{\mathcal{P}}_2(U,\mathfrak{v})$ involves a loss of central information, which may relevant beyond the perturbative level. Our approach has however the virtue of working for an arbitrary 2–term L_∞ algebra $\mathfrak{v}$ not necessarily arising from a differential Lie crossed module $(\mathfrak{g},\mathfrak{h})$.

Let us now assume that the global properties of the fields of a 2–term L_∞ algebra gauge theory with structure Lie algebra $\mathfrak{v}$ are defined by a 1–cell isomorphism class of 0–cells $g=\{g_{ij},\sigma_{ij},\Sigma_{ij},\tau_{ij}\}$ of $\check{\mathcal{P}}_2(U,\mathfrak{v})$.

The fact that the gluing data g_{ij} do not satisfy the standard 1–cocycle relation analogous to (3.8.5a) but the weaker condition (3.8.12g) is in general incompatible with the global single valuedness of the fields, by a well–known argument. As observed by Baez and Schreiber in [32,33], single valuedness is recovered imposing certain conditions involving simultaneosly the gluing data and the fields and ensuring that the relations

$$^{g_{ij}\diamond g_{jk}}\mathcal{F}_k = {}^{g_{ik}}\mathcal{F}_k, \qquad \text{on } U_{ijk}. \tag{3.8.27}$$

hold. Using crossed module notation, in which the 2–cell $W_{ijk} : g_{ik} \Rightarrow g_{ij} \diamond g_{jk}$ is represented as a pair (g_{ik}, W_{ijk}) with $W_{ijk} \in \mathrm{Gau}_2(U_{ijk}, \mathfrak{v})$ such that $s(W_{ijk}) = i$ and $t(W_{ijk}) = g_{ij} \diamond g_{jk} \diamond {g_{ik}}^{-1}$, (3.8.27) can be cast more compactly as

$$\mathcal{F}_i = {}^{t(W_{ijk})}\mathcal{F}_i, \qquad \text{on } U_{ijk}. \tag{3.8.28}$$

In general, the conditions do not take directly the form (3.8.27) or (3.8.28) but are conditions sufficient for these to hold emerging naturally in a higher categorical formulation of the theory.

Relation (3.8.8) describing the global matching of the local representations $\mathcal{F}_i$ of a field $\mathcal{F}$ is schematic and must be made more precise. In subsect. 3.2, we saw that, when M is diffeomorphic to $\mathbb{R}^d$, the fields of 2–term L_∞ algebra gauge theory organize in (dual) field doublets. In subsects. 3.4, we found further that the gauge transformation group acts naturally on doublets rather than their individual components. These properties should continue to hold in the appropriate form when M is a general d–fold. We are thus led to define a (dual) field doublet $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})$ on M to be a collection $\{({\mathcal{F}^{(1)}}_i, {\mathcal{F}^{(2)}}_i)\}$ of doublets $({\mathcal{F}^{(1)}}_i, {\mathcal{F}^{(2)}}_i)$ of the same type on the open sets U_i of the covering such that

$$ {\mathcal{F}^{(1)}}_i = {}^{g_{ij}}{\mathcal{F}^{(1)}}_j, \tag{3.8.29a}$$

$$ {\mathcal{F}^{(2)}}_i = {}^{g_{ij}}{\mathcal{F}^{(2)}}_j, \qquad \text{on } U_{ij}. \tag{3.8.29b}$$

In the concrete cases we have studied, this works out as follows.

A connection doublet (ω, Ω) on M is a collection $\{(\omega_i, \Omega_i)\}$ of connection doublets (ω_i, Ω_i) on the sets U_i such that (3.8.29) holds with $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}) = (\omega, \Omega)$, the right hand side being given by (3.4.7). Associated with the connection doublet, there is a curvature doublet (f, F) on M given locally as the collection $\{(f_i, F_i)\}$ of the curvature doublets (f_i, F_i) on the U_i and satisfying the matching relation (3.8.29) with $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}) = (f, F)$ and the right hand side given by (3.4.8).

Let a connection doublet (ω, Ω) on M be fixed. A bidegree (p, q) canonical

field doublet (ϕ, Φ) on M is a collection $\{(\phi_i, \Phi_i)\}$ of bidegree (p, q) canonical field doublets (ϕ_i, Φ_i) on the sets U_i such that (3.8.29) holds with $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}) = (\phi, \Phi)$, the right hand side being given by (3.4.9). In addition to the field doublet, we have a covariant derivative field doublet $(D\phi, D\Phi)$ on M given locally as the collection $\{(D\phi_i, D\Phi_i)\}$ of the covariant derivative field doublets $(D\phi_i, D\Phi_i)$ on the U_i and satisfying matching relations of the form (3.8.29) with $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}) = (D\phi, D\Phi)$ and the right hand side given by (3.4.10).

Similarly, a bidegree (r, s) dual field doublet (Υ, υ) on M is a collection $\{(\Upsilon_i, \upsilon_i)\}$ of bidegree (r, s) canonical dual field doublets (Υ_i, υ_i) on the sets U_i such that (3.8.29) holds with $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}) = (\Upsilon, \upsilon)$ and the right hand side given by (3.4.11). Further, we have a covariant derivative dual field doublet $(D\Upsilon, D\upsilon)$ on M given locally as the collection $\{(D\Upsilon_i, D\upsilon_i)\}$ of the covariant derivative dual field doublets $(D\Upsilon_i, D\upsilon_i)$ on the U_i and satisfying matching relations of the form (3.8.29) with $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)}) = (D\Upsilon, D\upsilon)$ and the right hand side given by (3.4.12).

Next, let us examine whether the above results can be adapted to the rectified set–up described in subsect. 3.5. Rectification requires gauge rectifiers. On a non trivial manifold M, a rectifier must be assigned on each open set of the chosen covering and appropriate matching relations must be satisfied for consistency. We find then that a 2–term L_∞ algebra gauge rectifier (λ, μ) on M is to be defined as a collection $\{(\lambda_i, \mu_i)\}$ of gauge rectifiers (λ_i, μ_i) on the sets U_i such that

$$\lambda_i(x, y) = {}^{g_{ij}}\lambda_j(x, y), \tag{3.8.30a}$$

$$\mu_i(x) = {}^{g_{ij}}\mu_j(x), \qquad \text{on } U_{ij}, \tag{3.8.30b}$$

the right hand side being given by (3.5.3). The compatibility of (3.8.12g) with the single valuedness of the rectifier requires that

$$\lambda_i(x, y) = {}^{t(W_{ijk})}\lambda_i(x, y), \tag{3.8.31a}$$

$$\mu_i(x) = {}^{t(W_{ijk})}\mu_i(x), \qquad \text{on } U_{ijk}, \tag{3.8.31b}$$

where crossed module notation is used, analogously to (3.8.28). It is likely that strong topologocal conditions must be satisfied in order for gauge rectifiers to exit. We leave the solution of this problem to future work.

With a gauge rectifier at one's disposal, it is possible to rectify canonical (dual) field doublets as well as define rectified covariant derivatives using eqs. (3.5.8), (3.5.9) and (3.5.10), (3.5.11), respectively. The analysis we have carried out for canonical (dual) field doublets and their covariant derivatives can be repeated almost *verbatim* also for their rectified counterparts. The compatibility of (3.8.12g) with the single valuedness of the rectified fields requires that (3.8.28) is still satisfied. Further, the matching relations have still the form (3.8.29), but the gauge transformations of the fields occurring in the right hand side are now given by (3.5.1), (3.5.2) for rectified doublets and similarly for their rectified covariant derivatives. Unlike in the non rectified case, all these matching relations are independent of any preassigned connection doublet (ω, Ω).

The above analysis can be generalized to more complicated situations, in which the fields do not group in doublets and are instead subject to more complex forms of gauge transformation involving several fields. It is only required that gauge transformation group action is left.

The results of this subsection provide us with the theoretical tools necessary for assessing whether a 2–term L_∞ algebra gauge theory defined on a d–fold M diffeomorphic to $\mathbb{R}^d$ can be defined globally also on a generic d–fold M: it is sufficient to check its gauge covariance. This requires the prior specification of the gauge transformation prescription of the fields, without which no statement about gauge covariance can be made.

Before concluding this subsection, an important remark is in order. Gauge covariance must not be confused with gauge symmetry. Gauge symmetry represents an objective property of a gauge theory, the invariance of the action and the observables under gauge symmetry variations of the fields. Gauge covariance,

instead, reflects the independence of a gauge theory from subjective frame choices and is manifest in the covariance of the basic equations under gauge transformation of the fields. Gauge symmetry variation is *active*: it does change the fields. Gauge transformation is *passive*: it does not, it simply governs the way the representations of the fields with respect to frames transform when the frames are changed. In many cases, gauge symmetry and gauge covariance are intimately related ed essentially equivalent (e. g. diffeomorphism symmetry and general covariance in general relativity), but this is not always so. These remarks should be kept in mind by the reader when studying the global properties of the 2–term L_∞ algebra gauge theories constructed in the next section.

3.9 Relation with other formulations

We conclude this subsection with the following remarks. In ordinary gauge theory, a connection on a principal G–bundle $P \to M$ is defined as a Lie algebra valued form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ satisfying the two Ehresmann conditions [65]. The first of these requires that ω equals the left invariant Maurer–Cartan form along the fibers of P, the second imposes that ω is G–equivariant. In an equivalent definition more suitable to our purposes, a connection ω is a differential graded commutative algebra morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(P)$, whose vertical projection $\mathrm{W}(\mathfrak{g}) \to \Omega_{\mathrm{vert}}{}^*(P)$ along the fibers of P is flat, and so it factors through $\mathrm{CE}(\mathfrak{g})$, and whose restriction to the invariant subalgebra $\ker d_{\mathrm{W}(\mathfrak{g})}\big|_{S(\mathfrak{g}^\vee[2])} \to \Omega^*(P)$ is basic and so it factors trough $\Omega^*(M)$ (cf. subsect. 3.1). When M is diffeomorphic to $\mathbb{R}^d$, $P \simeq M \times G$ and the second definition of connection boils down to the one we have given in subsect. 3.2 as a morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(M)$. When M is not, one has to pick a cover $U = \{U_i\}$ of M and define locally a connection as a morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(U_i)$ for each i. The resulting local data must then be assembled in a globally consistent way. The way of doing so is dictated by the topology of

the bundle P and codified in a G–valued cocycle. The problem with this classic approach is that it cannot be straightforwardly extended as it is to semistrict higher gauge theory, because there is no notion of total space of a principal 2–bundle that can be handled with the same ease. The authors of refs. [35, 36] tackle this problem reformulating ordinary gauge theory in a way that it can be directly generalized to the higher case. (See also the recent papers [68, 69]).

Given a Lie algebra $\mathfrak{g}$ and a Cartesian space $S = \mathbb{R}^d$, they consider the simplicial set of differential graded commutative algebra morphisms $\mathrm{W}(\mathfrak{g}) \to \Omega^*(S \times \Delta^k)$, where Δ^k is the standard geometric k–simplex with $k \geq 0$, whose vertical projection along Δ^k factors through $\mathrm{CE}(\mathfrak{g})$ and whose restriction to the invariant subalgebra $\ker d_{\mathrm{W}(\mathfrak{g})}\big|_{S(\mathfrak{g}^\vee[2])}$ factors through $\Omega^*(S)$. For each k, a morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(S \times \Delta^k)$ is equivalent to a connection 1–form ω on the trivial bundle $G \times (S \times \Delta^k)$, whose curvature 2–form f has components only along S. For $k = 0$, Δ^0 is the singleton 0 and a morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(S \times \Delta^0)$ as above reduces to an assignment of a connection 1–form ω on S, as we have already seen. For $k = 1$, Δ^1 is the 1–simplex $0 \to 1$ (an interval) and a morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(S \times \Delta^1)$ as above encodes a differential equation, which, once integrated, yields a gauge transformation g on S connecting the components of the connection 1–form ω along S at the extremes $0, 1$ of $0 \to 1$. For $k = 2$, Δ^2 is the 2–simplex $0 \to 1 \to 2$ (a triangle) and a morphism $\mathrm{W}(\mathfrak{g}) \to \Omega^*(S \times \Delta^2)$ identifies the gauge transformation $g_{21}g_{10}$ acting along the edges $0 \to 1$, $1 \to 2$ with that g_{20} acting along $0 \to 2$. On a non trivial manifold M equipped with a covering $U = \{U_i\}$, one considers morphisms with $S = U_i$, $S = U_{ij}$ and $S = U_{ijk}$ for $k = 0,\ 1\ 2$, respectively, which are compatible in the following sense: the restrictions of the resulting local data associated with the inclusions $U_{ijk} \subseteq U_{ij} \subseteq U_i$ coincide with the restrictions associated with the face inclusions $\Delta^0 \subset \Delta^1 \subset \Delta^2$. Then, those local data reduce to the familiar one defining a principal G–bundle bundle with connection: the Lie valued connection 1–forms ω_i and the group

valued transition functions g_{ij} satisfying the usual cocycle condition and relating ω_i, ω_j through gauge transformation. The advantage of this approach is that it generalizes directly to the higher case. One simply replaces the Lie algebra $\mathfrak{g}$ with a general L_∞ algebra $\mathfrak{v}$ (or even an L_∞ algebroid $\mathfrak{a}$) and goes through analogous steps.

A simplicial presheaf $\mathcal{G}$ is a presheaf over the category of Cartesian spaces CartSp such that for any Cartesian space $S \in \mathrm{CartSp}$, $\mathcal{G}(S) = \{\mathcal{G}_k(S)\}$ is a simplicial set. A simplicial presheaf is just (a presentation of) a smooth ∞–groupoid, an ∞ category in which k–morphisms are equivalences for all k with an appropriate notion of smoothness. For an L_∞ algebra $\mathfrak{v}$, the mapping $S \mapsto \{\mathrm{W}(\mathfrak{v}) \to \Omega^*(S \times \Delta^k)\}$, introduced in the previous paragraph, defines in the form of a simplicial presheaf a smooth ∞–groupoid $\exp(\mathfrak{v})_{\mathrm{conn}}$. The induced mapping $S \mapsto \{\mathrm{CE}(\mathfrak{v}) \to \Omega_{\mathrm{vert}}{}^*(S \times \Delta^k)\}$ defines a second smooth ∞–groupoid $\exp(\mathfrak{v})$, which may be viewed as the one that integrates $\mathfrak{v}$ in the sense of Lie theory. Replacing the Cartesian spaces S with the open sets of the nerve of a Čech covering $U = \{U_i\}$ of a manifold M defines an ∞ groupoid morphism from M, seen as an ∞ groupoid, to $\exp(\mathfrak{v})_{\mathrm{conn}}$ or $\exp(\mathfrak{v})$. The morphism encodes the set of local data defining an ∞ differential Čech cocycle describing an ∞ principal bundle P over M equipped with an ∞ connection ω, if $\exp(\mathfrak{v})_{\mathrm{conn}}$ is used, or the subset of data describing P alone, if $\exp(\mathfrak{v})$ is used instead. Note that we may view $\exp(\mathfrak{v})$ as the "structure" ∞ groupoid of P.

For a 2–term L_∞ algebra $\mathfrak{v}$, the method of refs. [35,36] yields the same definition of 2–connection on a trivial M we gave in subsect. 3.2 as a differential graded commutative algebra morphism $\mathrm{W}(\mathfrak{v}) \to \Omega^*(M)$. The approach to semistrict higher gauge theory, which we have described at length in this section, however differs from that of [35,36] in that it is "effective" in the following sense. We bypassed the difficulty of dealing with the ∞ groupoid $\exp(\mathfrak{v})$, which in its present abstract formulation does not lend itself easily to detailed calculations, and relied

instead on the automorphism 2–group $\mathrm{Aut}(\mathfrak{v})$. In this way we have avoided operating with gauge transformations as they are at their most basic level, roughly $\exp(\mathfrak{v})$–valued maps, and we have reduced ourselves to work with objects which somehow encapsulate the action of gauge transformations on fields in its most concrete form. We did this in subsects. 3.3, 3.4, where we defined a 2–term L_∞ algebra gauge transformation as an $\mathrm{Aut}_1(\mathfrak{v})$–valued map g plus a set of appended $\hat{\mathfrak{v}}_0$– and $\hat{\mathfrak{v}}_1$– and $\mathrm{Hom}(\hat{\mathfrak{v}}_0, \hat{\mathfrak{v}}_1)$–valued forms, σ_g, $\varSigma_g$ and $\tau_g(\cdot)$ respectively, satisfying certain relations and expressing the action of gauge transformation on fields in terms of these. Some guesswork was involved in this. We tested our approach in subsect. 3.6, where we checked that it reproduces the usual notions of gauge transformation when $\mathfrak{v}$ is strict. We were however unable to verify this property beyond the strict case. At any rate, we expect that the effective objects g, σ_g, $\varSigma_g$ and $\tau_g(\cdot)$ are expressible in terms of the more basic gauge transformations of the theory of [35,36], with the relations, which g, σ_g, $\varSigma_g$ and $\tau_g(\cdot)$ obey and which we treated as axioms, emerging as theorems. Our approach, though admittedly non fundamental, turns out to be quite efficient in the analysis of gauge covariance in field theoretic applications, as we shall show in the next section.

4 2–term L_∞ algebra BF gauge theory

In this section, we shall construct and analyze the semistrict Lie 2–algebra analog of the standard BF theory [26,27] providing in this way a simple but non trivial example of semistrict higher gauge theory. We shall first study the classical theory and, then, using an AKSZ approach [39], we shall work out the quantum theory and obtain by suitable gauge fixing a topological field theory.

Below, $\mathfrak{v}$ is the 2–term L_∞ structure algebra of the model and M is the oriented manifold on which fields propagate. M is taken 3–dimensional, because this is the simplest situation in which a 3–form curvature does not vanish identically.

4.1 Classical $\mathfrak{v}$ BF gauge theory

We consider first the case where M is diffeomorphic to $\mathbb{R}^3$ avoiding in this way the problems related to the global definedness of the theory.

The fields of classical $\mathfrak{v}$ BF gauge theory organize in a bidegree $(1,0)$ connection doublet $(\omega, \varOmega)$ and a further bidegree $(0,0)$ dual field doublet (B^+, b^+) (cf. sect. 3.2). Fields are supposed to fall off rapidly at the boundary of M, so that integration on M is convergent and integration by parts can be carried out without generating boundary contributions. The classical action of the theory is a rather straightforward generalization of that of standard BF gauge theory,

$$S_{\rm cl} = \int_M \Big[\langle b^+, f\rangle - \langle B^+, F\rangle\Big], \tag{4.1.1}$$

where the curvature doublet (f, F) is given by (3.2.11). The field equations are

$$f = 0, \tag{4.1.2a}$$

$$F = 0, \tag{4.1.2b}$$

$$DB^+ = 0, \tag{4.1.2c}$$

$$Db^+ = 0, \tag{4.1.2d}$$

where the covariant derivation D is defined in subsect. 3.2. They imply in particular that the connection doublet $(\omega, \varOmega)$ is flat, as in ordinary BF theory.

Classical $\mathfrak{v}$ BF gauge theory enjoys a high amount of gauge symmetry. The gauge symmetry variations of the fields are expressed in terms of ghost fields organized in a bidegree $(0,1)$ field doublet (c, C) and a bidegree $(-1,1)$ dual field doublet $(0, \beta^+)$,

$$\delta_{\rm cl}\omega = -Dc, \tag{4.1.3a}$$

$$\delta_{\rm cl}\varOmega = -DC, \tag{4.1.3b}$$

$$\delta_{\rm cl}B^+ = -[c, B^+]^\vee + \partial^\vee\beta^+, \tag{4.1.3c}$$

$$\delta_{\rm cl}b^+ = -[c, b^+]^\vee + [C, B^+]^\vee + [\omega, c, B^+]^\vee - D\beta^+. \tag{4.1.3d}$$

The action is invariant under the symmetry,

$$\delta_{\rm cl}S_{\rm cl} = 0, \tag{4.1.4}$$

as is easily verified. The expressions of the variations $\delta_{\rm cl}\omega$, $\delta_{\rm cl}\varOmega$ could be guessed on the basis of a formal similarity to that of the gauge field in standard gauge theory. Taking these for granted, the expressions of $\delta_{\rm cl}B^+$, $\delta_{\rm cl}b^+$ follow then from the requirement of invariance of the action.

For consistency, it should be possible to define gauge symmetry variations of the ghost fields rendering the gauge field variation operator $\delta_{\rm cl}$ nilpotent at least on–shell. These variations depend on the ghost field doublets (c, C), $(0, \beta^+)$ and a further ghost for ghost field seen as a bidegree $(-1,2)$ field doublet $(0, \varGamma)$,

$$\delta_{\rm cl}c = -\frac{1}{2}[c, c] + \partial\varGamma, \tag{4.1.5a}$$

$$\delta_{\rm cl}C = -[c, C] + \frac{1}{2}[\omega, c, c] - D\varGamma, \tag{4.1.5b}$$

$$\delta_{\rm cl}\beta^+ = -[c, \beta^+]^\vee + \frac{1}{2}[c, c, B^+]^\vee + [\varGamma, B^+]^\vee, \tag{4.1.5c}$$

$$\delta_{\rm cl}\varGamma = -[c, \varGamma] + \frac{1}{6}[c, c, c]. \tag{4.1.5d}$$

The expressions of the variations $\delta_{\rm cl}c$, $\delta_{\rm cl}C$, $\delta_{\rm cl}\beta^+$ with $\mathit{\Gamma}$ formally set to 0 may be inferred viewing c as akin to the standard gauge theory ghost. This suggests the first term in the right hand side of each of them. The remaining contributions are determined by the requirement of the on–shell nilpotency of $\delta_{\rm cl}$ upon taking into account that the bracket $[\cdot,\cdot]$ has a generally non trivial Jacobiator $[\cdot,\cdot,\cdot]$. The terms depending on $\mathit{\Gamma}$ reflect the existence of a gauge for gauge symmetry. The expression of $\delta_{\rm cl}\mathit{\Gamma}$ follows from a similar reasoning.

Using (4.1.3), (4.1.5), we find that ${\delta_{\rm cl}}^2\mathcal{F}=0$ for all fields and ghost fields $\mathcal{F}$ except for $\mathit{\Omega}$, b^+, in which case one has

$$ {\delta_{\rm cl}}^2\mathit{\Omega}=\frac{1}{2}[f,c,c]-[f,\mathit{\Gamma}], \tag{4.1.6a}$$

$$ {\delta_{\rm cl}}^2 b^+=\frac{1}{2}[c,c,DB^+]^\vee+[\mathit{\Gamma},DB^+]^\vee. \tag{4.1.6b}$$

Thus, on account of (4.1.2), $\delta_{\rm cl}$ is indeed nilpotent on shell, as required. We observe that the inclusion of the ghost for ghost $\mathit{\Gamma}$ is crucial for the nilpotency of $\delta_{\rm cl}$ in the ghost sector. Had $\mathit{\Gamma}$ not been there, ${\delta_{\rm cl}}^2$ would have vanished only up to a ghost field dependent gauge for gauge symmetry variation [14].

Since the gauge field variation operator $\delta_{\rm cl}$ is not nilpotent off–shell, the gauge symmetry algebra of the theory is open. Further, as the gauge symmetry admits a gauge for gauge symmetry, the gauge symmetry algebra is (at least) first stage

[14] If we had omitted $\mathit{\Gamma}$, we would have had in fact

$$ {\delta_{\rm cl}}^2 c=-\frac{1}{6}\partial([c,c,c]), \tag{4.1.7a}$$

$$ {\delta_{\rm cl}}^2 C=-\frac{1}{6}D([c,c,c]), \tag{4.1.7b}$$

$$ {\delta_{\rm cl}}^2 \beta^+=-\frac{1}{6}[[c,c,c],B^+]^\vee, \tag{4.1.7c}$$

where $(0,[c,c,c])$ is treated as a bidegree $(-1,3)$ field doublet. The right hand side of each of these relations is a gauge for gauge symmetry variation of the relevant ghost field with degree 3 parameter $-(1/6)[c,c,c]$. In gauge theory, indeed, one can expect $\delta_{\rm cl}$ to be nilpotent on–shell only up to gauge symmetry variations with ghost field dependent parameters.

reducible. These diseases will be cured by a suitable AKSZ reformulation of the model in the next subsection.

Let us now see whether $\mathfrak{v}$ BF gauge theory can be consistently formulated on a general closed 3–fold M. For reasons explained at the end of subsect. 3.8 [15], it is sufficient to check the gauge covariance of the model when M is diffeomorphic to $\mathbb{R}^3$. Recall that this requires that hypotheses on the the gauge transformation prescriptions of the fields and ghost fields be made.

Let $g \in \mathrm{Gau}_1(M, \mathfrak{v})$ be any 1–gauge transformation (cf. subsect. 3.3). We assume that the connection doublet (ω, Ω) transforms as in (3.4.7) and the dual field doublet (B^+, b^+) is canonical and, so, does as in (3.4.11). We then find that the classical Lagrangian $\mathcal{L}_{\mathrm{cl}}$ (the integrand of S_{cl} in (4.1.1)) is gauge invariant, ${}^g\mathcal{L}_{\mathrm{cl}} = \mathcal{L}_{\mathrm{cl}}$. Hence, the action S_{cl} can be defined also on a general 3–fold M.

Gauge covariance of the gauge symmetry field variations requires that ${}^g\delta_{\mathrm{cl}}\mathcal{F} = \delta_{\mathrm{cl}}{}^g\mathcal{F}$ for all fields and ghost fields $\mathcal{F}$ [16]. We assume that the ghost field doublet (c, C) is canonical and, so, transforms as in (3.4.9). Then, if we insist that the above property be satisfied in the ghost sector, the ghost dual field doublet $(0, \beta^+)$ and ghost for ghost field doublet $(0, \Gamma)$ cannot be canonical, but, instead, they must transform in a more complicated way, viz

$$ {}^g\beta^+ = g^\vee{}_0(\beta^+) - g^\vee{}_2(g_0(c), B^+), \tag{4.1.8a} $$

$$ {}^g\Gamma = g_1(\Gamma) - \frac{1}{2} g_2(c, c). \tag{4.1.8b} $$

We find then that ${}^g\delta_{\mathrm{cl}}\mathcal{F} = \delta_{\mathrm{cl}}{}^g\mathcal{F}$ for all fields and ghost fields $\mathcal{F}$ but Ω, b^+,

$$ {}^g\delta_{\mathrm{cl}}\Omega = \delta_{\mathrm{cl}}{}^g\Omega - g_2(c, f), \tag{4.1.9a} $$

$$ {}^g\delta_{\mathrm{cl}}b^+ = \delta_{\mathrm{cl}}{}^g b^+ - g^\vee{}_2(g_0(c), DB^+). \tag{4.1.9b} $$

---

[15] The reader is invited to keep in mind the remarks at the close of subsect. 3.8 below.

[16] ${}^g\delta_{\mathrm{cl}}\mathcal{F}$ is defined by replacing each occurrence of each field $\mathcal{G}$ in $\delta_{\mathrm{cl}}\mathcal{F}$ by ${}^g\mathcal{G}$.

Thus, the gauge field variation operator δ_{cl} is covariant only on shell. The gauge symmetry field variations, so, as given in (4.1.3), (4.1.5), cannot be defined on a general 3–fold M.

The failure of gauge symmetry field variation to be gauge covariant spoils the gauge symmetry invariance of the action S_{cl} when M is not diffeomorphic to $\mathbb{R}^3$. The verification of (4.1.4) requires the use of Stokes theorem to eliminate a term of the form $\int_M d\langle\beta^+, f\rangle$. On a non trivial M, this is legitimate only if $\langle\beta^+, f\rangle$ is gauge invariant. Unfortunately, this is not the case. Thus, as expected, (4.1.4) fails to hold for a general 3–fold M. This negative result can be traced back to the lack of gauge covariance of the gauge symmetry field variations. Indeed, it is easily verified that the combination $\langle\beta^+, f\rangle + \langle B^+, \delta_{\mathrm{cl}}\varOmega\rangle - \langle b^+, \delta_{\mathrm{cl}}\omega\rangle$ is gauge invariant. Therefore, the offending term causing the break down of the proof of (4.1.4) may be substituted by $-\int_M d(\langle B^+, \delta_{\mathrm{cl}}\varOmega\rangle - \langle b^+, \delta_{\mathrm{cl}}\omega\rangle)$, indicating that the origin of the problem is the gauge covariance failure of the variation $\delta_{\mathrm{cl}}\varOmega$. Again, this disease will be cured by the AKSZ reformulation of the model worked out in the next subsection.

4.2 AKSZ reformulation of $\mathfrak{v}$ BF gauge theory

AKSZ theory [39] is a method of constructing field theories satisfying the requirements of the BV quantization scheme [37,38] using solely the geometric data at hand. Following the AKSZ approach ensures that the resulting field theory can be consistently quantized on one hand and renders the whole construction in a way canonical on the other.

The basic steps of the AKSZ approach are the following.

1. The definition of a graded field space $\mathcal{F}$, the *BV field space*.

2. The assignment of a degree -1 symplectic form $\varOmega_{\mathrm{BV}}$ on $\mathcal{F}$, the *BV form*. Canonically associated with $\varOmega_{\mathrm{BV}}$ is a degree 1 Gerstenhaber bracket $(\cdot,\cdot)_{\mathrm{BV}}$ on

the functional algebra $\mathrm{Fun}(\mathcal{F})$ of $\mathcal{F}$, the *BV bracket*.

3. The construction of the appropriate degree 0 field functional S_{BV}, the *BV master action*, satisfying the *classical BV master equation*

$$(S_{\mathrm{BV}}, S_{\mathrm{BV}})_{\mathrm{BV}} = 0. \tag{4.2.1}$$

Then, according to BV theory, the field theory governed by S_{BV} is consistent and suitable for quantization.

Associated with S_{BV} is the *BV field variation operator* $\delta_{\mathrm{BV}} = (S_{\mathrm{BV}}, \cdot)_{\mathrm{BV}}$ on $\mathrm{Fun}(\mathcal{F})$. From (4.2.1), it follows that δ_{BV} is nilpotent

$${\delta_{\mathrm{BV}}}^2 = 0. \tag{4.2.2}$$

The cohomology of δ_{BV}, H^*_{BV}, is the theory's *BV cohomology*. Again from (4.2.1),

$$\delta_{\mathrm{BV}} S_{\mathrm{BV}} = 0. \tag{4.2.3}$$

The BV action S_{BV} is so BV invariant.

As anticipated in the previous subsection, the problems of $\mathfrak{v}$ BF gauge theory can be fixed through an AKSZ reformulation of the model. Next, we shall describe this in detail. Using the AKSZ method, we shall work out a field theory canonically associated to $\mathfrak{v}$, *AKSZ* $\mathfrak{v}$ *BF gauge theory*, and show that this is indeed the BV extension of $\mathfrak{v}$ BF gauge theory curing this latter's diseases.

The AKSZ construction is best implemented by using the *superfield formalism*, which we now briefly recall. For any vector space V, the space of $V[p]$–valued superfields is $\mathit{\Gamma}(T[1]M, V[p])$, where $T[1]M$ is the parity 1–shifted tangent bundle of M. In 3 dimensions, an element $\boldsymbol{\varphi} \in \mathit{\Gamma}(T[1]M, V[p])$ has a component expansion $\boldsymbol{\varphi} = \varphi^{0,p} + \varphi^{1,p-1} + \varphi^{2,p-2} + \varphi^{3,p-3}$, where $\varphi^{r,s}$ is a V–valued form degree r and ghost number degree s field. Superfields can be integrated on $T[1]M$. If $\boldsymbol{\varphi}$ is as above, then, by definition, $\int_{T[1]M} \varrho\, \boldsymbol{\varphi} = \int_M \varphi^{3,p-3}$, where ϱ is the standard supermeasure of $T[1]M$.

As usual, we first consider the case where M is diffeomorphic to $\mathbb{R}^3$. The field content of AKSZ $\mathfrak{v}$ BF gauge theory consists of superfields $\boldsymbol{p} \in \mathit{\Gamma}(T[1]M, \hat{\mathfrak{v}}_0{}^\vee[1])$, $\boldsymbol{q} \in \mathit{\Gamma}(T[1]M, \hat{\mathfrak{v}}_0[1])$, $\boldsymbol{P} \in \mathit{\Gamma}(T[1]M, \hat{\mathfrak{v}}_1{}^\vee[0])$, $\boldsymbol{Q} \in \mathit{\Gamma}(T[1]M, \hat{\mathfrak{v}}_1[2])$. The quadruple $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{Q})$ can be packaged into a superfield $\boldsymbol{\mathcal{A}} \in \mathit{\Gamma}(T[1]M, T^*[2](\hat{\mathfrak{v}}_0[1] \oplus \hat{\mathfrak{v}}_1[2]))$. Recall that $\hat{\mathfrak{v}}_0$, $\hat{\mathfrak{v}}_1$ are assumed conventionally to have ghost number degree 0.

The BV symplectic form of AKSZ $\mathfrak{v}$ BF gauge theory has the canonical form

$$\mathit{\Omega}_{\mathrm{BV}} = \int_{T[1]M} \varrho \Big[\langle \delta \boldsymbol{p}, \delta \boldsymbol{q} \rangle + \langle \delta \boldsymbol{P}, \delta \boldsymbol{Q} \rangle \Big]. \tag{4.2.4}$$

$\mathit{\Omega}_{\mathrm{BV}}$ is just the pull–back by $\boldsymbol{\mathcal{A}}$ of the canonical ghost number degree 2 symplectic form of $T^*[2](\hat{\mathfrak{v}}_0[1] \oplus \hat{\mathfrak{v}}_1[2])$. From BV theory, associated with $\mathit{\Omega}_{\mathrm{BV}}$ is the BV bracket $(\cdot, \cdot)_{\mathrm{BV}}$.

The BV action of AKSZ $\mathfrak{v}$ BF gauge theory is given by

$$\begin{aligned} S_{\mathrm{BV}} = \int_{T[1]M} \varrho \Big[& - \langle \boldsymbol{p}, \boldsymbol{d}\boldsymbol{q} - \frac{1}{2}[\boldsymbol{q}, \boldsymbol{q}] + \partial \boldsymbol{Q} \rangle \\ & + \langle \boldsymbol{P}, \boldsymbol{d}\boldsymbol{Q} - [\boldsymbol{q}, \boldsymbol{Q}] + \frac{1}{6}[\boldsymbol{q}, \boldsymbol{q}, \boldsymbol{q}] \rangle \Big]. \end{aligned} \tag{4.2.5}$$

S_{BV} satisfies the classical BV master equation (4.2.1) and, so, according to BV theory, the model is consistent and quantizable. The action S_{BV} is canonically associated with $\mathfrak{v}$, since the master equation is satisfied if and only if $\mathfrak{v} = (\mathfrak{v}_0, \mathfrak{v}_1, \partial, [\cdot, \cdot], [\cdot, \cdot, \cdot])$ is a 2–term L_∞ algebra (cf. subsect. 2.2).

The BV variations of the AKSZ $\mathfrak{v}$ BF gauge theory superfields are

$$\delta_{\mathrm{BV}} \boldsymbol{p} = \boldsymbol{d}\boldsymbol{p} - [\boldsymbol{q}, \boldsymbol{p}]^\vee + [\boldsymbol{Q}, \boldsymbol{P}]^\vee + \frac{1}{2}[\boldsymbol{q}, \boldsymbol{q}, \boldsymbol{P}]^\vee, \tag{4.2.6a}$$

$$\delta_{\mathrm{BV}} \boldsymbol{q} = \boldsymbol{d}\boldsymbol{q} - \frac{1}{2}[\boldsymbol{q}, \boldsymbol{q}] + \partial \boldsymbol{Q}, \tag{4.2.6b}$$

$$\delta_{\mathrm{BV}} \boldsymbol{P} = \boldsymbol{d}\boldsymbol{P} - [\boldsymbol{q}, \boldsymbol{P}]^\vee + \partial^\vee \boldsymbol{p}, \tag{4.2.6c}$$

$$\delta_{\mathrm{BV}} \boldsymbol{Q} = \boldsymbol{d}\boldsymbol{Q} - [\boldsymbol{q}, \boldsymbol{Q}] + \frac{1}{6}[\boldsymbol{q}, \boldsymbol{q}, \boldsymbol{q}]. \tag{4.2.6d}$$

Since S_{BV} solves the BV master equation (4.2.1), the BV superfield variation operator δ_{BV} satisfies (4.2.2), and so is nilpotent, as can also be directly verified

from (4.2.6). For the same reason, the BV action S_{BV} satisfies (4.2.3) and so is BV invariant.

Let us now see whether our AKSZ $\mathfrak{v}$ BF gauge theory can be consistently formulated on a general closed 3–fold M. Again, as explained in subsect. 3.8, it is enough to find the appropriate gauge transformation rules of the basic superfields and then check the model's gauge covariance for M diffeomorphic to $\mathbb{R}^3$.

For any 1–gauge transformation $g \in \mathrm{Gau}_1(M, \mathfrak{v})$, the appropriate expression of the gauge transformed superfields turns out to be

$$ {}^g\boldsymbol{p} = g^\vee{}_0(\boldsymbol{p} + \boldsymbol{\tau}_g{}^\vee(\boldsymbol{P})) - g^\vee{}_2(g_0(\boldsymbol{q} + \boldsymbol{\sigma}_g), \boldsymbol{P}), \tag{4.2.7a} $$

$$ {}^g\boldsymbol{q} = g_0(\boldsymbol{q} + \boldsymbol{\sigma}_g), \tag{4.2.7b} $$

$$ {}^g\boldsymbol{P} = g^\vee{}_1(\boldsymbol{P}), \tag{4.2.7c} $$

$$ {}^g\boldsymbol{Q} = g_1(\boldsymbol{Q} - \boldsymbol{\Sigma}_g - \boldsymbol{\tau}_g(\boldsymbol{q} + \boldsymbol{\sigma}_g)) - \frac{1}{2} g_2(\boldsymbol{q} + \boldsymbol{\sigma}_g, \boldsymbol{q} + \boldsymbol{\sigma}_g), \tag{4.2.7d} $$

where $\boldsymbol{\sigma}_g$, $\boldsymbol{\Sigma}_g$, $\boldsymbol{\tau}_g$ are just σ_g, Σ_g, τ_g seen as one–component superfields (cf. subsect. 3.3). (4.2.7) can be justified as follows. The usual connection doublet (ω, Ω) must appear in the component expansion of the superfield pair $(-\boldsymbol{q}, \boldsymbol{Q})$, as is evident by matching of their form and ghost number degrees, the minus sign being conventional. So, the pair $(-\boldsymbol{q}, \boldsymbol{Q})$ must transform as if it were a connection doublet (cf. eqs. (3.4.7)). This explains the form of (4.2.7b), (4.2.7d). Similarly, to have gauge covariance, $(-\boldsymbol{P}, \boldsymbol{p})$ must transform as a bidegree $(0, 0)$ canonical dual field doublet (cf. eqs. (3.4.11)), yielding (4.2.7a), (4.2.7c).

Let $\boldsymbol{\Lambda}_{\mathrm{BV}}$ be the integrand superfield of Ω_{BV} in (4.2.4). It is easy to check that $\boldsymbol{\Lambda}_{\mathrm{BV}}$ is gauge invariant, ${}^g\boldsymbol{\Lambda}_{\mathrm{BV}} = \boldsymbol{\Lambda}_{\mathrm{BV}}$. Hence, the BV form Ω_{BV} can be defined also on a general 3–fold M.

Let $\boldsymbol{\mathcal{L}}_{\mathrm{BV}}$ be the BV Lagrangian, the integrand superfield of S_{BV} in (4.2.5). A simple calculation shows that $\boldsymbol{\mathcal{L}}_{\mathrm{BV}}$ is gauge invariant, ${}^g\boldsymbol{\mathcal{L}}_{\mathrm{BV}} = \boldsymbol{\mathcal{L}}_{\mathrm{BV}}$. Hence, the BV action S_{BV}, also, can be defined on a general 3–fold M.

In the verification of the BV master equation (4.2.1), one uses Stokes' theorem to eliminate a term of the form $2\int_{T[1]M}\boldsymbol{d}\boldsymbol{\mathcal{L}}_{\mathrm{BV}}$. This is legitimate also on a non trivial M, as ${}^g\boldsymbol{\mathcal{L}}_{\mathrm{BV}}=\boldsymbol{\mathcal{L}}_{\mathrm{BV}}$. So, the master equation holds on any 3–fold M.

Gauge covariance of the BV field variations (4.2.6) requires that ${}^g\delta_{\mathrm{BV}}\boldsymbol{\mathcal{F}}=\delta_{\mathrm{BV}}{}^g\boldsymbol{\mathcal{F}}$ for all superfields $\boldsymbol{\mathcal{F}}$ (cf. fn. 16). This is expected by the gauge covariance of the BV master equation (4.2.1) and it can be directly verified using (4.2.6), (4.2.7). So, the field variations (4.2.6) are globally defined on a general 3–fold M.

We can now show that AKSZ $\mathfrak{v}$ BF gauge theory is the appropriate BV extension of classical $\mathfrak{v}$ BF gauge theory curing all the diseases of this latter. To do so, we expand the basic superfields $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ in components

$$\boldsymbol{p}=-\beta^{+}+b^{+}-\omega^{+}+c^{+}, \tag{4.2.8a}$$

$$\boldsymbol{q}=c-\omega+b-\beta, \tag{4.2.8b}$$

$$\boldsymbol{P}=-B^{+}+\mathit{\Omega}^{+}-C^{+}+\mathit{\Gamma}^{+}, \tag{4.2.8c}$$

$$\boldsymbol{Q}=\mathit{\Gamma}-C+\mathit{\Omega}-B, \tag{4.2.8d}$$

where the terms in the right hand side are written down in increasing order of form degree and decreasing order of ghost number degree and the choice of the signs of the component fields is conventional [17]. We then write down the BV action and the BV superfield variations in terms of the components by substituting the expansions (4.2.8) into (4.2.5) and (4.2.6), respectively. The resulting expressions are rather messy and are collected in app. A for the interested reader. However, it is not difficult to see that the truncation of the BV action (4.2.5) to the ghost

[17] Here, ϕ^{+} is the antifield of ϕ. In order to have the component fields organized naturally in doublets as in subsect. 3.2, we have not followed the convention, common in BV theory, of requiring the antifields to have negative ghost number degree. In AKSZ theory, this is allowed since the field/antifield splitting is simply a conventional choice of local Darboux coordinates for the BV form in BV field space. A redefinition of such separation is just a BV field symplectomorphism leaving the BV form invariant.

number degree 0 fields ω, $\varOmega$, B^+, b^+ reproduces precisely the classical action of $\mathfrak{v}$ BF gauge theory given in eq. (4.1.1). Further, the truncation of the BV variations of the ghost number degree 0 fields ω, $\varOmega$, B^+, b^+ to the ghost number degree 0, 1 fields ω, $\varOmega$, B^+, b^+, c, C, β^+ reproduces precisely the corresponding classical gauge symmetry variations given in eqs. (4.1.3). Similarly, we obtain the classical gauge symmetry variations of the ghost number degree 1, 2 fields c, C, β^+, $\varGamma$ given in eqs. (4.1.5) by truncating the BV variations of those fields to ghost number degree 0, 1, 2 fields ω, $\varOmega$, B^+, b^+, c, C, β^+, $\varGamma$.

AKSZ $\mathfrak{v}$ BF gauge theory, however, is not affected by the problems plaguing classical $\mathfrak{v}$ BF gauge theory: unlike its classical counterpart $\delta_{\rm cl}$, the BV field variation operator $\delta_{\rm BV}$ is nilpotent off–shell and gauge covariant. There is nevertheless a cost for this gain. The component fields ω, $\varOmega$ no longer behave under gauge transformation as the components of a connection doublet (cf. eqs. (3.4.7)), but mix with the component fields b, c. This renders the geometrical interpretation of the component fields less evident.

4.3 Rectified AKSZ $\mathfrak{v}$ BF gauge theory

The non linear nature of the superfield gauge transformations (4.2.7) in AKSZ $\mathfrak{v}$ BF gauge theory makes it difficult to control gauge covariance and carry out gauge fixing. It is possible to reformulate the theory in an equivalent way that is covariant under a completely linear rectified gauge transformation action. We call the resulting model *rectified AKSZ* $\mathfrak{v}$ *BF gauge theory*. The price for this is that the BV action and field variations are definitely more complicated.

The field content of rectified AKSZ $\mathfrak{v}$ BF gauge theory consists of four superfields $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ of the same type as that of plain AKSZ $\mathfrak{v}$ BF gauge theory. The BV symplectic form is given again by (4.2.4).

The BV action of rectified AKSZ $\mathfrak{v}$ BF gauge theory is given instead by an

expression different from (4.2.1) involving a 2–term L_∞ algebra gauge rectifier (λ, μ) and a background connection doublet $(\bar{\omega}, \bar{\Omega})$ with curvature doublet $(\bar{f}, \bar{F})$ (cf. subsects. 3.2, 3.5). Explicitly, the action is given by

$$S_{\mathrm{BV}} = \int_{T[1]M} \varrho \Big[-\langle \boldsymbol{p}, -\bar{\boldsymbol{f}}_{\lambda,\mu} + \bar{\boldsymbol{D}}_{\lambda,\mu}\boldsymbol{q} - \frac{1}{2}[\boldsymbol{q},\boldsymbol{q}]_\lambda + \partial\boldsymbol{Q}\rangle \tag{4.3.1}$$
$$+ \langle \boldsymbol{P}, \bar{\boldsymbol{F}}_{\lambda,\mu} + \bar{\boldsymbol{D}}_{\lambda,\mu}\boldsymbol{Q} - [\boldsymbol{q}, \boldsymbol{Q} + \bar{\boldsymbol{\Omega}} - \frac{1}{2}\boldsymbol{\lambda}(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}) + \boldsymbol{\mu}(\bar{\boldsymbol{\omega}})]_\lambda$$
$$+ \frac{1}{6}[\boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}]_\lambda + \frac{1}{6}[\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}]_\lambda$$
$$+ \frac{1}{2}\boldsymbol{v}_{\lambda,\mu}(\boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}) - \frac{1}{2}\boldsymbol{v}_{\lambda,\mu}(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}) + \boldsymbol{w}_{\lambda,\mu}(\boldsymbol{q})\rangle\Big],$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{v}_{\lambda,\mu}$, $\boldsymbol{w}_{\lambda,\mu}$, $\bar{\boldsymbol{\omega}}$, $\bar{\boldsymbol{\Omega}}$, $\bar{\boldsymbol{f}}_{\lambda,\mu}$, $\bar{\boldsymbol{F}}_{\lambda,\mu}$, $\bar{\boldsymbol{D}}_{\lambda,\mu}$ are just λ, μ, $v_{\lambda,\mu}$, $w_{\lambda,\mu}$, $\bar{\omega}$, $\bar{\Omega}$, $\bar{f}_{\lambda,\mu}$, $\bar{F}_{\lambda,\mu}$, $D_{\lambda,\mu}$ seen as one–component superfields. Above, the deformed brackets $[\cdot,\cdot]_\lambda$, $[\cdot,\cdot,\cdot]_\lambda$ are defined in (3.5.4). The derived rectifiers $v_{\lambda,\mu}$, $w_{\lambda,\mu}$ are defined in (3.5.6). $\bar{f}_{\lambda,\mu}$, $\bar{F}_{\lambda,\mu}$ are the rectified counterpart of the curvature components $\bar{f}$, $\bar{F}$ and are defined according to (3.5.8), viz $\bar{f}_{\lambda,\mu} = \bar{f}$, $\bar{F}_{\lambda,\mu} = \bar{F} + \lambda(\bar{\omega}, \bar{f}) - \mu(\bar{f})$. The rectified covariant derivative $D_{\lambda,\mu}$ is defined according to (3.5.10), (3.5.11) with ω replaced by $\bar{\omega}$. Again, S_{BV} satisfies the classical BV master equation (4.2.1) and, so, the rectified model is consistent and quantizable.

The BV variations of the rectified $\mathfrak{v}$ AKSZ BF gauge theory superfields are

$$\delta_{\mathrm{BV}}\boldsymbol{p} = \bar{\boldsymbol{D}}_{\lambda,\mu}\boldsymbol{p} - [\boldsymbol{q},\boldsymbol{p}]_\lambda{}^\vee + [\boldsymbol{Q} + \bar{\boldsymbol{\Omega}} - \frac{1}{2}\boldsymbol{\lambda}(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}) + \boldsymbol{\mu}(\bar{\boldsymbol{\omega}}), \boldsymbol{P}]_\lambda{}^\vee \tag{4.3.2a}$$
$$+ \frac{1}{2}[\boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{P}]_\lambda{}^\vee + \boldsymbol{v}_{\lambda,\mu}{}^\vee(\boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{P}) + \boldsymbol{w}_{\lambda,\mu}{}^\vee(\boldsymbol{P}),$$

$$\delta_{\mathrm{BV}}\boldsymbol{q} = -\bar{\boldsymbol{f}}_{\lambda,\mu} + \bar{\boldsymbol{D}}_{\lambda,\mu}\boldsymbol{q} - \frac{1}{2}[\boldsymbol{q},\boldsymbol{q}]_\lambda + \partial\boldsymbol{Q}, \tag{4.3.2b}$$

$$\delta_{\mathrm{BV}}\boldsymbol{P} = \bar{\boldsymbol{D}}_{\lambda,\mu}\boldsymbol{P} - [\boldsymbol{q},\boldsymbol{P}]_\lambda{}^\vee + \partial^\vee\boldsymbol{p}, \tag{4.3.2c}$$

$$\delta_{\mathrm{BV}}\boldsymbol{Q} = \bar{\boldsymbol{F}}_{\lambda,\mu} + \bar{\boldsymbol{D}}_{\lambda,\mu}\boldsymbol{Q} - [\boldsymbol{q}, \boldsymbol{Q} + \bar{\boldsymbol{\Omega}} - \frac{1}{2}\boldsymbol{\lambda}(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}) + \boldsymbol{\mu}(\bar{\boldsymbol{\omega}})]_\lambda \tag{4.3.2d}$$
$$+ \frac{1}{6}[\boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}]_\lambda + \frac{1}{6}[\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}]_\lambda$$
$$+ \frac{1}{2}\boldsymbol{v}_{\lambda,\mu}(\boldsymbol{q}-\bar{\boldsymbol{\omega}}, \boldsymbol{q}-\bar{\boldsymbol{\omega}}) - \frac{1}{2}\boldsymbol{v}_{\lambda,\mu}(\bar{\boldsymbol{\omega}},\bar{\boldsymbol{\omega}}) + \boldsymbol{w}_{\lambda,\mu}(\boldsymbol{q}).$$

Above, the deformed cobrackets $[\cdot,\cdot]_\lambda{}^\vee$, $[\cdot,\cdot,\cdot]_\lambda{}^\vee$ are defined according to the same prescription as their undeformed counterparts (cf. eqs. (3.2.16b)–(3.2.16e)). The derived gauge corectifier $(v_{\lambda,\mu}{}^\vee, w_{\lambda,\mu}{}^\vee)$ are defined by the relations $\langle\Xi, v_{\lambda,\mu}(x,y)\rangle = -\langle v_{\lambda,\mu}{}^\vee(x,\Xi), y\rangle$, $\langle\Xi, w_{\lambda,\mu}(x)\rangle = -\langle w_{\lambda,\mu}{}^\vee(\Xi), y\rangle$. As before, δ_{BV} satisfies (4.2.2), and is thus nilpotent, and S_{BV} satisfies (4.2.3), and is thus BV invariant.

In rectified AKSZ $\mathfrak{v}$ BF gauge theory, the gauge transformation action is no longer given by (4.2.7) but, instead, takes the fully linear rectified form

$$ {}^g\boldsymbol{p} = g^\vee{}_0(\boldsymbol{p}) \tag{4.3.3a} $$

$$ {}^g\boldsymbol{q} = g_0(\boldsymbol{q}), \tag{4.3.3b} $$

$$ {}^g\boldsymbol{P} = g^\vee{}_1(\boldsymbol{P}), \tag{4.3.3c} $$

$$ {}^g\boldsymbol{Q} = g_1(\boldsymbol{Q}), \tag{4.3.3d} $$

for $g \in \mathrm{Gau}_1(M,\mathfrak{v})$.

It can be verified that the integrand superfields in the rectified analog of (4.2.4) and (4.3.1) are gauge invariant. So, for reasons explained in subsect. 3.8, the rectified BV form Ω_{BV} and action S_{BV} can be defined also on a general 3–fold M. One also finds that the rectified BV superfield variations (4.3.2) are gauge covariant, ensuring that they also can be defined globally on a general 3–fold M.

Non rectified and rectified AKSZ $\mathfrak{v}$ BF gauge theories are related by an invertible superfield redefinition depending on the gauge rectifier (λ,μ) mapping the fields of the former into those of the latter,

$$ \boldsymbol{p}_{\mathrm{r}} = \boldsymbol{p}_{\mathrm{nr}} - \boldsymbol{\lambda}^\vee(\boldsymbol{q}_{\mathrm{nr}}, \boldsymbol{P}_{\mathrm{nr}}) - \boldsymbol{\mu}^\vee(\boldsymbol{P}_{\mathrm{nr}}), \tag{4.3.4a} $$

$$ \boldsymbol{q}_{\mathrm{r}} = \boldsymbol{q}_{\mathrm{nr}} + \bar{\boldsymbol{\omega}}, \tag{4.3.4b} $$

$$ \boldsymbol{P}_{\mathrm{r}} = \boldsymbol{P}_{\mathrm{nr}}, \tag{4.3.4c} $$

$$ \boldsymbol{Q}_{\mathrm{r}} = \boldsymbol{Q}_{\mathrm{nr}} - \frac{1}{2}\boldsymbol{\lambda}(\boldsymbol{q}_{\mathrm{nr}}, \boldsymbol{q}_{\mathrm{nr}}) - \boldsymbol{\mu}(\boldsymbol{q}_{\mathrm{nr}}) - \bar{\boldsymbol{\Omega}} + \frac{1}{2}\boldsymbol{\lambda}(\bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\omega}}) - \boldsymbol{\mu}(\bar{\boldsymbol{\omega}}). \tag{4.3.4d} $$

Above, the subscript r and nr mark the rectified and non rectified version of the

superfields, respectively. The gauge corectifier $(\lambda^\vee, \mu^\vee)$ is defined below eq. (3.5.9). Remarkably, (4.3.4) is BV canonical field map: it transforms the BV symplectic form $\Omega_{\rm BV}$ of the non rectified theory into that of the rectified one (cf. eq. (4.2.4) and its rectified counterpart). It also maps the BV action $S_{\rm BV}$ and the BV superfield variations of the non rectified theory (cf. eqs. (4.2.5), (4.2.6)) into those of the rectified one (cf. eqs. (4.3.1), (4.3.2)). Last but not least, it maps the non rectified superfields gauge transforming according to (4.2.7) into their rectified counterparts gauge transforming instead according to (4.3.3). The non rectified and the rectified theories are therefore fully equivalent. So, since the former is independent from the gauge rectifier (λ, μ) and the background connection doublet $(\bar{\omega}, \bar{\Omega})$, the latter also is, in spite of appearances.

Rectified AKSZ $\mathfrak{v}$ BF gauge theory can be written in terms of superfield components. The component expansions of the superfields are again given by eqs. (4.2.8), but now the components organize in rectified (dual) field doublets. The component expressions of the BV action and field variations are reported in app. B for the interested reader.

The advantage of using the rectified version of AKSZ $\mathfrak{v}$ BF gauge theory will become clear in the procedure of gauge fixing of the theory, which is the topic of the next subsection.

4.4 Gauge fixing of AKSZ $\mathfrak{v}$ BF gauge theory

In BV field theory, gauge fixing is implemented by restricting the theory's field configurations to lie on a suitable Lagrangian submanifold $\mathcal{L}$ of BV field space $\mathcal{F}$ [37, 38]. The generating functional Ψ of $\mathcal{L}$ is called *gauge fermion*. By definition, $\mathcal{L}$ is determined by Ψ as the locus in $\mathcal{F}$ where ${\phi_i}^+ = \delta_r\Psi/\delta\phi_i$ holds, where the ϕ_i, ${\phi_i}^+$ are the appropriate set of fields/antifields and $\delta_r/\delta\phi_i$ denotes right functional differentiation.

In general, for a given BV field theory, the construction of Ψ is not possible without suitably modifying the theory. Extra fields and their antifields must be added to $\mathcal{F}$. These in turn introduce further contributions to the BV odd symplectic form Ω_{BV} and master action S_{BV}. In the extended BV field theory so obtained, Ψ can be built and the gauge fixing can be carried out by restricting to the associated field space Lagrangian submanifold $\mathcal{L}$ as indicated above.

The extra fields and antifields required by the construction of the gauge fermion Ψ mentioned in the previous paragraph organize in trivial pairs. For each gauge fixing condition, a trivial pair of fields and the trivial pair of their antifields is needed. The nature of the trivial pairs is determined by the form of gauge fixing conditions and the requirement that Ψ has ghost number degree -1.

In more detail, Ψ is constructed through the following procedure.

Let V be a vector space. A bidegree (m,n) V *trivial pair* is a pair of fields (ϕ,ψ) with $\phi\in\Omega^m(M,V^\vee[n])$, $\psi\in\Omega^m(M,V^\vee[n+1])$. The antifields of the fields of the pair (ϕ,ψ) form a bidegree $(d-m,-n-2)$ $V^\vee$ trivial pair (ψ^+,ϕ^+), the *antipair* of (ϕ,ψ), where $d=\dim M$ (in our case $d=3$).

The BV odd symplectic form Ω_{tBV} of the trivial pair (ϕ,ψ), (ψ^+,ϕ^+) system has the canonical form

$$\Omega_{\mathrm{tBV}}=\int_M\Big[\langle\delta\phi^+,\delta\phi\rangle_{V^\vee}+\langle\delta\psi^+,\delta\psi\rangle_{V^\vee}\Big]. \tag{4.4.1}$$

The BV action S_{tBV} of the (ϕ,ψ), (ψ^+,ϕ^+) system is

$$S_{\mathrm{tBV}}=\int_M\langle\psi,\phi^+\rangle_V \tag{4.4.2}$$

and is easily seen to verify the BV master equation (4.2.1). The BV field variations of the pair fields are given by

$$\delta_{\mathrm{tBV}}\phi=(-1)^{m+n}\psi,\qquad \delta_{\mathrm{tBV}}\psi=0, \tag{4.4.3a}$$

$$\delta_{\mathrm{tBV}}\psi^+=-\phi^+,\qquad \delta_{\mathrm{tBV}}\phi^+=0. \tag{4.4.3b}$$

As usual, δ_{tBV} satisfies (4.2.2), and is thus nilpotent, and S_{tBV} satisfies (4.2.3), and is thus BV invariant. When several trivial pairs and their antipairs are involved, the BV form Ω_{tBV} is again of the form (4.4.1) with a contribution from each pair/antipair and similarly for the BV action S_{tBV} in eq. (4.4.2).

If a gauge fixing condition is of the form $\varpi = 0$, where ϖ is a bidegree (p, q) V–valued field expression, one needs a bidegree $(d-p, -q-1)$ V trivial pair (ϕ, ψ) together with its bidegree $(p, q-1)$ $V^\vee$ trivial antipair (ψ^+, ϕ^+). The corresponding contribution to the gauge fermion Ψ is then given by

$$\Psi_\varpi = \int_M \langle \phi, \varpi \rangle_V. \tag{4.4.4}$$

There is one such contribution for each gauge fixing condition ϖ.

The reason why, instead of relying on the plain AKSZ gauge theory of subsect. 4.2, we endeavoured to formulate its rectified version in subsect. 4.3 is that gauge fixing is carried out in a manifestly gauge covariant manner far more easily in the latter than the former. In fact, the gauge fixing conditions involve the single components of the superfields and this makes controlling gauge covariance problematic in the non rectified theory where the components mix under gauge transformation, while it poses no problem in the rectified one, where they do not (cf. subsects. 4.2, 4.3).

In AKSZ $\mathfrak{v}$ BF gauge theory, a gauge fixing condition is required for every field of positive form degree acted upon by the exterior differential d. From (4.3.1) and (4.2.8b), (4.2.8d), we need then a condition for each of the components ω, b, β, C, Ω, B, which we choose to be of the form

$$\bar{D}_{\lambda,\mu} * \omega = 0, \tag{4.4.5a}$$

$$\bar{D}_{\lambda,\mu} * b = 0, \tag{4.4.5b}$$

$$* \beta = 0, \tag{4.4.5c}$$

$$\bar{D}_{\lambda,\mu} * C = 0, \tag{4.4.5d}$$

$$\bar{D}_{\lambda,\mu} * \varOmega = 0, \tag{4.4.5e}$$

$$* B = 0, \tag{4.4.5f}$$

where $*$ is the Hodge star operator associated with some metric h on M. (4.4.5a), (4.4.5b), (4.4.5d), (4.4.5e) are standard Lorenz gauge fixing conditions. (4.4.5c), (4.4.5f) are background gauge fixing conditions. The reason why the gauge fixing prescriptions of β, B are stronger than those of ω, b, C, $\varOmega$ is that $d\beta = 0$, $dB = 0$ identically in $d = 3$ dimensions. Proceeding as explained above and taking into account the form of the conditions (4.4.5), we introduce three $\hat{\mathfrak{v}}_0$ trivial pairs $(\tilde{c}, \tilde{\theta})$, $(\tilde{\omega}, \tilde{\xi})$, $(\tilde{q}, \tilde{\chi})$ of bidegree $(0, -1)$, $(1, 0)$, $(3, 1)$ together with their $\hat{\mathfrak{v}}_0{}^\vee$ trivial antipairs $(\tilde{\theta}^+, \tilde{c}^+)$, $(\tilde{\xi}^+, \tilde{\omega}^+)$, $(\tilde{\chi}^+, \tilde{q}^+)$ of bidegree $(3, -1)$, $(2, -2)$, $(0, -3)$ for the conditions (4.4.5a)–(4.4.5c) and three $\hat{\mathfrak{v}}_1$ trivial pairs $(\tilde{\varGamma}, \tilde{H})$, $(\tilde{C}, \tilde{\varTheta})$, $(\tilde{F}, \tilde{\varSigma})$ of bidegree $(0, -2)$, $(1, -1)$, $(3, 0)$ together with their $\hat{\mathfrak{v}}_1{}^\vee$ trivial antipairs $(\tilde{H}^+, \tilde{\varGamma}^+)$, $(\tilde{\varTheta}^+, \tilde{C}^+)$, $(\tilde{\varSigma}^+, \tilde{F}^+)$ of bidegree $(3, 0)$, $(2, -1)$, $(0, -2)$ for the conditions (4.4.5d)–(4.4.5f), respectively. The above gauge fixing is however not sufficient. The resulting gauge fermion $\varPsi$ contains a portion $\int_M [\langle \tilde{\omega}, \bar{D}_{\lambda,\mu} * b \rangle + \langle \tilde{C}, \bar{D}_{\lambda,\mu} * \varOmega \rangle]$ whereby two of the constrains which define the field space Lagrangian submanifold $\mathcal{L}$ take the form $b^+ = *D_{\lambda,\mu}\tilde{\omega}$ and $\varOmega^+ = *D_{\lambda,\mu}\tilde{C}$. As $\tilde{\omega}$, $\tilde{C}$ have positive form degree and are acted upon by d, two more gauge fixing conditions are required,

$$\bar{D}_{\lambda,\mu} * \tilde{\omega} = 0, \tag{4.4.6a}$$

$$\bar{D}_{\lambda,\mu} * \tilde{C} = 0. \tag{4.4.6b}$$

We thus introduce one $\hat{\mathfrak{v}}_0{}^\vee$ trivial pair (a, θ) of bidegree $(0, -1)$ together with its $\hat{\mathfrak{v}}_0$ trivial antipair (θ^+, a^+) of bidegree $(3, -1)$ for the gauge fixing conditions (4.4.6a) and one $\hat{\mathfrak{v}}_1{}^\vee$ trivial pair $(\varPhi, H)$ of bidegree $(0, 0)$ together with its $\hat{\mathfrak{v}}_0$ trivial antipair $(H^+, \varPhi^+)$ of bidegree $(3, -2)$ for the gauge fixing conditions (4.4.6b), respectively. The trivial pair BV action and field variations can now be written down easily using (4.4.2) and (4.4.3).

From (4.4.4), the gauge fermion associated with the above gauge fixing is

$$\Psi = \int_M \Big[\langle \tilde{c}, \bar{D}_{\lambda,\mu} * \omega\rangle + \langle \tilde{\omega}, \bar{D}_{\lambda,\mu} * b\rangle + \langle \tilde{q}, *\beta\rangle + \langle \tilde{\Gamma}, \bar{D}_{\lambda,\mu} * C\rangle \tag{4.4.7}$$

$$+ \langle \tilde{C}, \bar{D}_{\lambda,\mu} * \Omega\rangle + \langle \tilde{F}, *B\rangle + \langle a, \bar{D}_{\lambda,\mu} * \tilde{\omega}\rangle + \langle \Phi, \bar{D}_{\lambda,\mu} * \tilde{C}\rangle\Big].$$

The constraints defining the associated BV field space Lagrangian submanifold $\mathcal{L}$ are written down explicitly in app. C.

The gauge fixed action is now given by $I = (S_{\rm BV} + S_{\rm tBV})|_{\mathcal{L}}$. Explicitly

$$I = \int_M \Big[\langle *\bar{D}_{\lambda,\mu}\tilde{\omega}, \bar{f}_{\lambda,\mu} + \bar{D}_{\lambda,\mu}\omega + \frac{1}{2}[\omega,\omega]_\lambda + [c,b]_\lambda - \partial\Omega\rangle \tag{4.4.8}$$

$$- \langle *\tilde{F}, \bar{F}_{\lambda,\mu} + \bar{D}_{\lambda,\mu}\Omega + [\omega, \Omega + \bar{\Omega} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda$$

$$+ [c, B]_\lambda + [b, C]_\lambda + [\beta, \Gamma]_\lambda - [\omega + \bar{\omega}, c, b]_\lambda - \frac{1}{2}[c, c, \beta]_\lambda$$

$$- \frac{1}{6}[\omega + \bar{\omega}, \omega + \bar{\omega}, \omega + \bar{\omega}]_\lambda + \frac{1}{6}[\bar{\omega}, \bar{\omega}, \bar{\omega}]_\lambda + v_{\lambda,\mu}(c, b)$$

$$+ \frac{1}{2}v_{\lambda,\mu}(\omega + \bar{\omega}, \omega + \bar{\omega}) - \frac{1}{2}v_{\lambda,\mu}(\bar{\omega},\bar{\omega}) - w_{\lambda,\mu}(\omega)\rangle$$

$$- \langle *\bar{D}_{\lambda,\mu}\tilde{c}, \bar{D}_{\lambda,\mu}c + [\omega, c]_\lambda - \partial C\rangle - \langle *\bar{D}_{\lambda,\mu}\tilde{C}, \bar{D}_{\lambda,\mu}C + [\omega, C]_\lambda$$

$$+ [c, \Omega + \bar{\Omega} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda + [b, \Gamma]_\lambda - \frac{1}{2}[c, c, b]_\lambda$$

$$- \frac{1}{2}[\omega + \bar{\omega}, \omega + \bar{\omega}, c]_\lambda + v_{\lambda,\mu}(\omega + \bar{\omega}, c) - w_{\lambda,\mu}(c)\rangle +$$

$$+ \langle *\bar{D}_{\lambda,\mu}\tilde{\Gamma}, \bar{D}_{\lambda,\mu}\Gamma + [\omega, \Gamma]_\lambda + [c, C]_\lambda - \frac{1}{2}[\omega + \bar{\omega}, c, c]_\lambda$$

$$+ \frac{1}{2}v_{\lambda,\mu}(c, c)\rangle - \langle *\tilde{q}, \bar{D}_{\lambda,\mu}b + [\omega, b]_\lambda + [c, \beta]_\lambda - \partial B\rangle$$

$$- \langle \tilde{\theta}, \bar{D}_{\lambda,\mu} * \omega\rangle - \langle \tilde{\xi}, \bar{D}_{\lambda,\mu} * b + *\bar{D}_{\lambda,\mu}a\rangle + \langle \tilde{\chi}, *\beta\rangle$$

$$+ \langle \tilde{H}, \bar{D}_{\lambda,\mu} * C\rangle + \langle \tilde{\Theta}, \bar{D}_{\lambda,\mu} * \Omega - *\bar{D}_{\lambda,\mu}\Phi\rangle - \langle \tilde{\Sigma}, *B\rangle$$

$$- \langle \theta, \bar{D}_{\lambda,\mu} * \tilde{\omega}\rangle + \langle H, \bar{D}_{\lambda,\mu} * \tilde{C}\rangle\Big].$$

The BRST field variation operator is $s = (\delta_{\rm BV} + \delta_{\rm tBV})|_{\mathcal{L}}$. One finds so

$$sc = -\frac{1}{2}[c, c]_\lambda + \partial\Gamma, \tag{4.4.9a}$$

$$s\omega = -\bar{D}_{\lambda,\mu}c - [\omega, c]_\lambda + \partial C, \tag{4.4.9b}$$

$$sb = -\bar{f}_{\lambda,\mu} - \bar{D}_{\lambda,\mu}\omega - \frac{1}{2}[\omega,\omega]_\lambda - [c,b]_\lambda + \partial\Omega, \tag{4.4.9c}$$

$$s\beta = -\bar{D}_{\lambda,\mu}b - [\omega,b]_\lambda - [c,\beta]_\lambda + \partial B, \tag{4.4.9d}$$

$$s\Gamma = -[c,\Gamma]_\lambda + \frac{1}{6}[c,c,c]_\lambda, \tag{4.4.9e}$$

$$sC = -\bar{D}_{\lambda,\mu}\Gamma - [\omega,\Gamma]_\lambda - [c,C]_\lambda + \frac{1}{2}[\omega+\bar{\omega},c,c]_\lambda - \frac{1}{2}v_{\lambda,\mu}(c,c), \tag{4.4.9f}$$

$$\begin{aligned} s\Omega = {} & -\bar{D}_{\lambda,\mu}C - [\omega,C]_\lambda - [c,\Omega+\bar{\Omega} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda \\ & - [b,\Gamma]_\lambda + \frac{1}{2}[c,c,b]_\lambda + \frac{1}{2}[\omega+\bar{\omega},\omega+\bar{\omega},c]_\lambda \\ & - v_{\lambda,\mu}(\omega+\bar{\omega},c) + w_{\lambda,\mu}(c), \end{aligned} \tag{4.4.9g}$$

$$\begin{aligned} sB = {} & -\bar{F}_{\lambda,\mu} - \bar{D}_{\lambda,\mu}\Omega - [\omega,\Omega+\bar{\Omega} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda \\ & - [c,B]_\lambda - [b,C]_\lambda - [\beta,\Gamma]_\lambda + [\omega+\bar{\omega},c,b]_\lambda + \frac{1}{2}[c,c,\beta]_\lambda \\ & + \frac{1}{6}[\omega+\bar{\omega},\omega+\bar{\omega},\omega+\bar{\omega}]_\lambda - \frac{1}{6}[\bar{\omega},\bar{\omega},\bar{\omega}]_\lambda - v_{\lambda,\mu}(c,b) \\ & - \frac{1}{2}v_{\lambda,\mu}(\omega+\bar{\omega},\omega+\bar{\omega}) + \frac{1}{2}v_{\lambda,\mu}(\bar{\omega},\bar{\omega}) + w_{\lambda,\mu}(\omega), \end{aligned} \tag{4.4.9h}$$

$$s\tilde{c} = -\tilde{\theta}, \tag{4.4.9i}$$

$$s\tilde{\theta} = 0, \tag{4.4.9j}$$

$$s\tilde{C} = \tilde{\Theta}, \tag{4.4.9k}$$

$$s\tilde{\Theta} = 0, \tag{4.4.9l}$$

$$s\Phi = H, \tag{4.4.9m}$$

$$sH = 0, \tag{4.4.9n}$$

$$s\tilde{\Gamma} = \tilde{H}, \tag{4.4.9o}$$

$$s\tilde{H} = 0, \tag{4.4.9p}$$

$$s\tilde{\omega} = -\tilde{\xi}, \tag{4.4.9q}$$

$$s\tilde{\xi} = 0, \tag{4.4.9r}$$

$$sa = -\theta, \tag{4.4.9s}$$

$$s\theta = 0, \tag{4.4.9t}$$

$$s\tilde{F} = -\tilde{\Sigma}, \tag{4.4.9u}$$

$$s\tilde{\Sigma} = 0, \tag{4.4.9v}$$

$$s\tilde{q} = \tilde{\chi}, \tag{4.4.9w}$$

$$s\tilde{\chi} = 0. \tag{4.4.9x}$$

The BRST field variation operator s is nilpotent off–shell

$$s^2 = 0. \tag{4.4.10}$$

This property of s is fundamental. There is no need to directly verify it: its holding is obvious a priori. From the component expansions (4.2.8) and the expressions of the BV field variations (4.3.2), (4.4.3a), it is apparent that fields are closed under the action of the BV field variation operators $\delta_{\rm BV}$, $\delta_{\rm tBV}$, unlike antifields. Thus, s reduces simply to the restriction of $\delta_{\rm BV}$, $\delta_{\rm tBV}$ to the field sector of the extended BV field space. The nilpotence of s is therefore an immediate consequence of that of $\delta_{\rm BV}$, $\delta_{\rm tBV}$.

The BRST invariant action I is simply related to the gauge fermion Ψ,

$$I = s\Psi. \tag{4.4.11}$$

From (4.4.10), (4.4.11), it is then immediate that

$$sI = 0. \tag{4.4.12}$$

The action I is thus BRST invariant as required.

Let us now pose to discuss the results just obtained. The use of rectified fields avoids field mixing under gauge transformation and renders all the gauge fixing conditions manifestly gauge covariant. It also makes the trivial pair BV action $S_{\rm tBV}$ and the gauge fermion Ψ and, consequently, also the gauge fixed action I and the BRST field variation operator s manifestly gauge invariant. In this way,

on a non trivial 3–fold M, rectified fields behave much as sections of ordinary vector bundles. Further, S_{tBV}, Ψ, I and s, once expressed in terms of rectified fields, are all manifestly globally defined. In the non rectified theory, the same could also be achieved of course, but at the price of dealing with fields mixing under gauge transformation in a complicated way and, so, loosing manifest gauge covariance. On a non trivial 3–fold M, non rectified fields belong to a rather non standard non linear geometry in which the fields' global properties are not describable in the familiar terms of ordinary vector bundle geometry (cf. subsect. 3.8). Further, S_{tBV}, Ψ, I and s, when expressed in terms of non rectified fields, are not manifestly globally defined. On the other hand, certain expressions are simpler in non rectified form. Derectification can be achieved, if one wishes so, by substituting the expressions (4.3.4) in component form in the above relations.

4.5 Topological $\mathfrak{v}$ BF gauge theory

By (4.4.11), the gauge fixed AKSZ $\mathfrak{v}$ BF gauge theory is topological. We call this theory *topological* $\mathfrak{v}$ *BF gauge theory* because it stems from classical $\mathfrak{v}$ BF gauge theory introduced in subsect. 4.1.

In topological field theory, topological operators are of particular salience, because their correlators are independent from all background fields which enter only in the gauge fermion and so compute topological invariants of the background geometry. The topological operators are the BRST cohomology classes. A thorough discussion of the BRST cohomology of topological $\mathfrak{v}$ BF gauge theory is therefore in order. This requires the prior study of the BV cohomology of AKSZ $\mathfrak{v}$ BF gauge theory, beginning with the non rectified version of the model and then proceeding to the rectified one.

In subsect. 3.1, we have seen that the 2–term L_∞ algebra $\mathfrak{v}$ is characterized by the cohomology $H^*_{CE}(\mathfrak{v})$ of the Chevalley–Eilenberg cochain complex

$(\mathrm{CE}(\mathfrak{v}), \mathcal{Q}_{\mathrm{CE}(\mathfrak{v})})$, where $\mathrm{CE}(\mathfrak{v}) = S(\mathfrak{v}^\vee[1])$ and $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ is defined in (3.1.8).

Assume first that M is diffeomorphic to $\mathbb{R}^3$. The algebra $\Omega^*(M, \mathrm{CE}(\mathfrak{v}))$ of $\mathrm{CE}(\mathfrak{v})$–valued forms is graded by the total form plus $\mathrm{CE}(\mathfrak{v})$ degree. Further, $\Omega^*(M, \mathrm{CE}(\mathfrak{v}))$ admits a natural coboundary operator given by

$$d_{\mathrm{CE}(\mathfrak{v})} = d - \mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}, \tag{4.5.1}$$

where d is the exterior differential. So, $(\Omega^*(M, \mathrm{CE}(\mathfrak{v})), d_{\mathrm{CE}(\mathfrak{v})})$ is a cochain complex, with which there is associated a cohomology $H^*_{dCE}(M, \mathrm{CE}(\mathfrak{v}))$.

Let $\varkappa \in \Omega^n(M, \mathrm{CE}(\mathfrak{v}))$. Then, $\varkappa$ has an expansion

$$\varkappa = \sum_{p,q \geq 0} \frac{1}{p!q!} \varkappa_{p,q}(\pi, \ldots, \pi; \mathit{\Pi}, \ldots, \mathit{\Pi}), \tag{4.5.2}$$

where $\varkappa_{p,q}$ is a $n - p - 2q$ form with values in $\bigwedge^p \hat{\mathfrak{v}}_0{}^\vee \otimes \bigvee^q \hat{\mathfrak{v}}_1{}^\vee$ (see subsect. 2.2). $\varkappa_{p,q}$ vanishes unless $n - d, 0 \leq p + 2q \leq n$, where $d = 3$ in our case. In non rectified AKSZ $\mathfrak{v}$ BF gauge theory, we associate with $\varkappa$ the degree n superfield

$$\boldsymbol{\mathcal{O}}_\varkappa = \sum_{p,q \geq 0} \frac{1}{p!q!} \boldsymbol{\varkappa}_{p,q}(\boldsymbol{p}, \ldots, \boldsymbol{p}; \boldsymbol{P}, \ldots, \boldsymbol{P}), \tag{4.5.3}$$

where $\boldsymbol{\varkappa}_{p,q}$ is $\varkappa_{p,q}$ seen as a superfield. A simple calculation shows that

$$\delta_{\mathrm{BV}} \boldsymbol{\mathcal{O}}_\varkappa = \boldsymbol{d} \boldsymbol{\mathcal{O}}_\varkappa - \boldsymbol{\mathcal{O}}_{d_{\mathrm{CE}(\mathfrak{v})}\varkappa}. \tag{4.5.4}$$

Thus, if $\varkappa$ is a $d_{\mathrm{CE}(\mathfrak{v})}$–cocycle, $d_{\mathrm{CE}(\mathfrak{v})}\varkappa = 0$, then

$$\delta_{\mathrm{BV}} \boldsymbol{\mathcal{O}}_\varkappa = \boldsymbol{d} \boldsymbol{\mathcal{O}}_\varkappa. \tag{4.5.5}$$

$\boldsymbol{\mathcal{O}}_\varkappa$ is then δ_{BV}–closed mod $\boldsymbol{d}$. Further, when $\varkappa$ is a $d_{\mathrm{CE}(\mathfrak{v})}$–coboundary, $\varkappa = d_{\mathrm{CE}(\mathfrak{v})}\vartheta$ for some $\vartheta \in \Omega^{n-1}(M, \mathrm{CE}(\mathfrak{v}))$, then

$$\boldsymbol{\mathcal{O}}_\varkappa = \boldsymbol{d} \boldsymbol{\mathcal{O}}_\vartheta - \delta_{\mathrm{BV}} \boldsymbol{\mathcal{O}}_\vartheta. \tag{4.5.6}$$

$\boldsymbol{\mathcal{O}}_\varkappa$ is then δ_{BV}–exact mod $\boldsymbol{d}$. The algorithm followed here, so, produces only mod $\boldsymbol{d}$ BV cohomology classes, while we want to obtain true BV cohomology classes. The way of achieving this is well–established.

A supercycle (superboundary) $\boldsymbol{\gamma}$ is a formal sum of ordinary cycles (boundaries), $\boldsymbol{\gamma} = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$. Superfields can be integrated on supercycles. If $\boldsymbol{\varphi} = \varphi^0 + \varphi^1 + \varphi^2 + \varphi^3$ is a superfield expanded in its components and $\boldsymbol{\gamma}$ is a supercycle, then we have $\int_{\boldsymbol{\gamma}} \boldsymbol{\varphi} = \sum_{r=1}^{3} \int_{\gamma_r} \phi^r$. By Stokes theorem, we have $\int_{\boldsymbol{\gamma}} \boldsymbol{d\varphi} = 0$. For any supercycle $\boldsymbol{\gamma}$, we define

$$\langle \boldsymbol{\gamma}, \boldsymbol{\mathcal{O}}_{\varkappa} \rangle = \int_{\boldsymbol{\gamma}} \boldsymbol{\mathcal{O}}_{\varkappa}. \tag{4.5.7}$$

From (4.5.5), when $\varkappa$ is a cocycle,

$$\delta_{\mathrm{BV}} \langle \boldsymbol{\gamma}, \boldsymbol{\mathcal{O}}_{\varkappa} \rangle = 0. \tag{4.5.8}$$

Further, from (4.5.6), when $\varkappa$ is a coboundary,

$$\langle \boldsymbol{\gamma}, \boldsymbol{\mathcal{O}}_{\varkappa} \rangle = -\delta_{\mathrm{BV}} \langle \boldsymbol{\gamma}, \boldsymbol{\mathcal{O}}_{\vartheta} \rangle. \tag{4.5.9}$$

It follows that, for a fixed supercycle $\boldsymbol{\gamma}$, the mapping $\varkappa \mapsto \langle \boldsymbol{\gamma}, \boldsymbol{\mathcal{O}}_{\varkappa} \rangle$ is a homomorphism of the cohomology $H^*_{dCE}(M, \mathrm{CE}(\mathfrak{v}))$ into the model's BV cohomology H^*_{BV}.

We require the gauge invariance of the superfields $\boldsymbol{\mathcal{O}}_{\varkappa}$. An action of $\mathrm{Gau}_1(M, \mathfrak{v})$ on $\Omega^*(M, \mathrm{CE}(\mathfrak{v}))$ must be defined such that, for $g \in \mathrm{Gau}_1(M, \mathfrak{v})$,

$$\boldsymbol{\mathcal{O}}_{{}^g\varkappa}({}^g\boldsymbol{p}; {}^g\boldsymbol{P}) = \boldsymbol{\mathcal{O}}_{\varkappa}(\boldsymbol{p}; \boldsymbol{P}), \tag{4.5.10}$$

where we have explicitly indicated the dependence of $\boldsymbol{\mathcal{O}}_{\varkappa}$ on the basic superfields $\boldsymbol{p}$, $\boldsymbol{P}$ for clarity and the action of $\mathrm{Gau}_1(M, \mathfrak{v})$ on $\boldsymbol{p}$, $\boldsymbol{P}$ is given by (4.2.7b), (4.2.7d). Given the non linear nature of the gauge transformation action on superfields, a simple explicit expression of ${}^g\varkappa$ cannot be written. However, it is easy to see that ${}^g\varkappa$ is linear in $\varkappa$ [18].

When M is a non trivial 3–fold, the above analysis must be modified as fol-

[18] Indeed, by (4.5.2) and (4.5.10), one has $\boldsymbol{\mathcal{O}}_{{}^g(a_1\varkappa_1 + a_2\varkappa_2)}({}^g\boldsymbol{p}; {}^g\boldsymbol{P}) = \boldsymbol{\mathcal{O}}_{a_1\varkappa_1 + a_2\varkappa_2}(\boldsymbol{p}; \boldsymbol{P}) = a_1\boldsymbol{\mathcal{O}}_{\varkappa_1}(\boldsymbol{p}; \boldsymbol{P}) + a_2\boldsymbol{\mathcal{O}}_{\varkappa_2}(\boldsymbol{p}; \boldsymbol{P}) = a_1\boldsymbol{\mathcal{O}}_{{}^g\varkappa_1}({}^g\boldsymbol{p}; {}^g\boldsymbol{P}) + a_2\boldsymbol{\mathcal{O}}_{{}^g\varkappa_2}({}^g\boldsymbol{p}; {}^g\boldsymbol{P}) = \boldsymbol{\mathcal{O}}_{a_1{}^g\varkappa_1 + a_2{}^g\varkappa_2}({}^g\boldsymbol{p}; {}^g\boldsymbol{P})$, which implies that ${}^g(a_1\varkappa_1 + a_2\varkappa_2) = a_1{}^g\varkappa_1 + a_2{}^g\varkappa_2$

lows. In accordance with the general discussion of subsect. 3.8, fields are defined only locally in M and the matching of the local representation of the fields on the sets of an open covering $\{U_i\}$ of M is governed by suitable matching data $\{g_{ij}\}$ with $g_{ij} \in \mathrm{Gau}_1(U_{ij}, \mathfrak{v})$ acting on the representations by gauge transformation. The construction of globally defined BV cohomology classes described above requires then that $\varkappa$ be replaced by a collection $\{\varkappa_i\}$ with $\varkappa_i \in \Omega^n(U_i, \mathrm{CE}(\mathfrak{v}))$ such that $\varkappa_i = {}^{g_{ij}}\varkappa_j$ on U_{ij} and $\varkappa_i = {}^{t(W_{ijk})}\varkappa_i$ on U_{ijk} (cf. eqs. (3.8.8) and (3.8.28)). Let $E^n(\mathfrak{v})$ be the linear space of all $\mathrm{CE}(\mathfrak{v})$–valued n–form local data satisfying these matching conditions. The locally defined coboundary operator $d_{\mathrm{CE}(\mathfrak{v})}$ extends to a globally defined one $d_{\mathrm{CE}(\mathfrak{v})}$ on $E^n(M, \mathfrak{v})$ [19]. Thus, $(E^*(M, \mathfrak{v}), d_{\mathrm{CE}(\mathfrak{v})})$ is a cochain complex. Proceeding as earlier, one establishes a homomorphism $H_{d_{\mathrm{CE}(\mathfrak{v})}}{}^*(E^*(M, \mathfrak{v}))$ into H^*_{BV}.

The BV cohomology classes we have obtained in this way in the non rectified version of AKSZ $\mathfrak{v}$ BF gauge theory, can be transposed with little effort to the rectified one, expressing the non rectified superfields in terms of the rectified ones by inverting (4.3.4b), (4.3.4d),

$$\boldsymbol{q}_{\mathrm{nr}} = \boldsymbol{q}_{\mathrm{r}} - \bar{\boldsymbol{\omega}}, \tag{4.5.11a}$$

$$\boldsymbol{Q}_{\mathrm{nr}} = \boldsymbol{Q}_{\mathrm{r}} + \frac{1}{2}\boldsymbol{\lambda}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{r}}) + \boldsymbol{\mu}(\boldsymbol{q}_{\mathrm{r}}) - \boldsymbol{\lambda}(\bar{\boldsymbol{\omega}}, \bar{\boldsymbol{q}}_{\mathrm{r}}) + \bar{\boldsymbol{\Omega}}. \tag{4.5.11b}$$

The expressions of the representatives of the rectified theory's BV cohomology classes so obtained depend on the gauge rectifier (λ, μ) and the background connection doublet $(\bar{\omega}, \bar{\Omega})$. This dependence is however an artifact of the reparametri-

[19] This can be seen as follows. Take M diffeomorphic to $\mathbb{R}^3$. Let $\varkappa \in \Omega^n(M, \mathrm{CE}(\mathfrak{v}))$ and let $g \in \mathrm{Gau}_1(M, \mathfrak{v})$. Since $\delta_{\mathrm{BV}}{}^g\boldsymbol{p} = {}^g\delta_{\mathrm{BV}}\boldsymbol{p}$, $\delta_{\mathrm{BV}}{}^g\boldsymbol{P} = {}^g\delta_{\mathrm{BV}}\boldsymbol{P}$ (cf. subsect. 4.2), we have $\delta_{\mathrm{BV}}\boldsymbol{\mathcal{O}}_{\varkappa}(\boldsymbol{p};\boldsymbol{P}) = \delta_{\mathrm{BV}}\boldsymbol{\mathcal{O}}_{{}^g\varkappa}({}^g\boldsymbol{p};{}^g\boldsymbol{P}) = \boldsymbol{d}\boldsymbol{\mathcal{O}}_{{}^g\varkappa}({}^g\boldsymbol{p};{}^g\boldsymbol{P}) - \boldsymbol{\mathcal{O}}_{d_{\mathrm{CE}(\mathfrak{v})}{}^g\varkappa}({}^g\boldsymbol{p};{}^g\boldsymbol{P})$ from (4.5.4), (4.5.10). But we also have $\delta_{\mathrm{BV}}\boldsymbol{\mathcal{O}}_{\varkappa}(\boldsymbol{p};\boldsymbol{P}) = \boldsymbol{d}\boldsymbol{\mathcal{O}}_{\varkappa}(\boldsymbol{p};\boldsymbol{P}) - \boldsymbol{\mathcal{O}}_{d_{\mathrm{CE}(\mathfrak{v})}\varkappa}(\boldsymbol{p};\boldsymbol{P}) = \boldsymbol{d}\boldsymbol{\mathcal{O}}_{{}^g\varkappa}({}^g\boldsymbol{p};{}^g\boldsymbol{P}) - \boldsymbol{\mathcal{O}}_{{}^g d_{\mathrm{CE}(\mathfrak{v})}\varkappa}({}^g\boldsymbol{p};{}^g\boldsymbol{P})$ again by (4.5.4), (4.5.10). Hence, $d_{\mathrm{CE}(\mathfrak{v})}{}^g\varkappa = {}^g d_{\mathrm{CE}(\mathfrak{v})}\varkappa$. So, on a non trivial 3–fold M, where $\varkappa \in E^n(M, \mathfrak{v})$ has only local representations matching by gauge transformation as indicated above, $d_{\mathrm{CE}(\mathfrak{v})}\varkappa$ is globally defined and $d_{\mathrm{CE}(\mathfrak{v})}\varkappa \in E^{n+1}(M, \mathfrak{v})$.

zation (4.5.11). For reasons explained at the end of subsect. 4.3, the non rectified and rectified versions are fully equivalent and, so, their BV cohomologies are isomorphic. Since the cohomology of the former is independent from (λ, μ), $(\bar{\omega}, \bar{\Omega})$, that of the latter also is.

As explained in subsect. 4.4, the BRST field variation operator s is simply the restriction of $\delta_{\rm BV}$, $\delta_{\rm tBV}$ to the field sector of the extended BV field space of the rectified theory. Since the component fields of the superfields $\boldsymbol{p}$, $\boldsymbol{P}$ belong to that sector, the representatives of BV cohomology classes we have constructed are automatically also representatives of BRST cohomology classes. They thus represent topological operators of topological $\mathfrak{v}$ BF gauge theory. By standard arguments of topological field theory, their correlators are independent from the background data (λ, μ), $(\bar{\omega}, \bar{\Omega})$ and h and, so, compute topological invariants.

4.6 $\mathfrak{v}$ Chern–Simons gauge theory

In this final subsection, we shall describe briefly the 2–term L_∞ algebra analog of standard Chern–Simons (CS) theory [44]. It is another example of 2–term L_∞ algebra gauge theory and is related closely to the BF gauge theory we have studied above in great detail. A more thorough analysis of the CS model will be presented elsewhere.

$\mathfrak{v}$ CS gauge theory can be defined when $\mathfrak{v}$ is a reduced 2–term L_∞ algebra equipped with an invariant metric. A 2–term L_∞ algebra $\mathfrak{v} = (\mathfrak{v}_0, \mathfrak{v}_1, \partial, [\cdot,\cdot], [\cdot,\cdot,\cdot])$ is said *reduced* if $\ker\partial = 0$. In that case $\mathfrak{v}_1$ can be considered as a subspace of $\mathfrak{v}_0$ and the indication of ∂ can be omitted. An *invariant metric* on a 2–term L_∞ algebra $\mathfrak{v} = (\mathfrak{v}_0, \mathfrak{v}_1, \partial, [\cdot,\cdot], [\cdot,\cdot,\cdot])$ is a non singular symmetric bilinear map $(\cdot,\cdot) : \mathfrak{v}_0 \vee \mathfrak{v}_0 \mapsto \mathfrak{v}_0$ with the invariance properties $(x, [z, y]) + ([z, x], y) = 0$ and $(x, [w, z, y]) + ([w, z, x], y) = 0$.

The base manifold of $\mathfrak{v}$ CS gauge theory is a 4–fold N, which we take to

be diffeomorphic to $\mathbb{R}^4$ to avoid for the time being the subtleties of the global definition of the model. The fields of classical $\mathfrak{v}$ CS gauge theory constitute a bidegree $(1,0)$ connection doublet $(\omega, \varOmega)$. The classical action is

$$S_{cl} = \int_N \Big[-(\varOmega, f + \frac{1}{2}\varOmega) + \frac{1}{24}(\omega, [\omega, \omega, \omega]) \Big]. \tag{4.6.1}$$

The field equations of the theory read

$$f = 0, \tag{4.6.2a}$$

$$F = 0. \tag{4.6.2b}$$

They imply that the connection doublet $(\omega, \varOmega)$ is flat, as in standard CS theory and reproduce the field equations (4.1.2a), (4.1.2b) of classical $\mathfrak{v}$ BF gauge theory.

Like BF theory, classical $\mathfrak{v}$ CS gauge theory enjoys a high amount of gauge symmetry. The gauge symmetry variations of the fields are expressed in terms of ghost fields organized in a bidegree $(0,1)$ field doublet (c, C),

$$\delta_{\rm cl}\omega = -Dc, \tag{4.6.3a}$$

$$\delta_{\rm cl}\varOmega = -DC. \tag{4.6.3b}$$

They coincide in form with the gauge symmetry variations of the corresponding fields of classical $\mathfrak{v}$ BF gauge theory (cf. eqs. (4.1.3a), (4.1.3b)). As is straightforward to verify, the action is invariant under the symmetry,

$$\delta_{\rm cl}S_{\rm cl} = 0. \tag{4.6.4}$$

As in BF theory, it should be possible to define gauge symmetry variations of the ghost fields rendering the gauge field variation operator $\delta_{\rm cl}$ nilpotent at least on–shell. These variations depend on the ghost field doublet (c, C) and a further ghost for ghost field seen as a bidegree $(-1,2)$ field doublet $(0, \varGamma)$,

$$\delta_{\rm cl}c = -\frac{1}{2}[c, c] + \varGamma, \tag{4.6.5a}$$

$$\delta_{\rm cl}C = -[c, C] + \frac{1}{2}[\omega, c, c] - D\varGamma, \tag{4.6.5b}$$

$$\delta_{\mathrm{cl}} \Gamma = -[c, \Gamma] + \frac{1}{6}[c, c, c]. \tag{4.6.5c}$$

Again, they coincide in form with the gauge symmetry variations of the corresponding fields of classical $\mathfrak{v}$ BF gauge theory (cf. eqs, (4.1.5a), (4.1.5b), (4.1.5d)). Using (4.6.3), (4.6.5), we find that ${\delta_{\mathrm{cl}}}^2 \mathcal{F} = 0$ for all fields and ghost fields $\mathcal{F}$ except for Ω, in which case one has

$${\delta_{\mathrm{cl}}}^2 \Omega = \frac{1}{2}[f, c, c] - [f, \Gamma]. \tag{4.6.6}$$

Again, as in BF theory, δ_{cl} is nilpotent but only on–shell by the field equation (4.6.2a) (cf. eq. (4.1.6a)).

As $\mathfrak{v}$ BF gauge theory, $\mathfrak{v}$ CS gauge theory admits an AKSZ formulation, *AKSZ $\mathfrak{v}$ CS gauge theory*. The field content of this consists of a $\hat{\mathfrak{v}}_0[1]$–valued superfield $\boldsymbol{q}$ and a $\hat{\mathfrak{v}}_1[2]$–valued superfield $\boldsymbol{Q}$. The BV symplectic form is given by

$$\Omega_{\mathrm{BV}} = \int_{T[1]N} \varrho(\delta \boldsymbol{q}, \delta \boldsymbol{Q}). \tag{4.6.7}$$

Then, by BV theory, associated with Ω_{BV} is the BV bracket $(\cdot, \cdot)_{\mathrm{BV}}$.

The BV action AKSZ $\mathfrak{v}$ CS gauge theory is

$$S_{\mathrm{BV}} = \int_{T[1]N} \varrho\Big[(\boldsymbol{Q}, \boldsymbol{d}\boldsymbol{q} - \frac{1}{2}[\boldsymbol{q}, \boldsymbol{q}] + \frac{1}{2}\boldsymbol{Q}) + \frac{1}{24}(\boldsymbol{q}, [\boldsymbol{q}, \boldsymbol{q}, \boldsymbol{q}])\Big]. \tag{4.6.8}$$

It is straightforward to verify that S_{BV} satisfies the classical BV master equation (4.2.1) and, so, according to BV theory, the model is consistent and quantizable.

The BV variations of the AKSZ $\mathfrak{v}$ CS gauge theory superfields are

$$\delta_{\mathrm{BV}} \boldsymbol{q} = \boldsymbol{d}\boldsymbol{q} - \frac{1}{2}[\boldsymbol{q}, \boldsymbol{q}] + \boldsymbol{Q}, \tag{4.6.9a}$$

$$\delta_{\mathrm{BV}} \boldsymbol{Q} = \boldsymbol{d}\boldsymbol{Q} - [\boldsymbol{q}, \boldsymbol{Q}] + \frac{1}{6}[\boldsymbol{q}, \boldsymbol{q}, \boldsymbol{q}]. \tag{4.6.9b}$$

They agree in form with the BV variations of the corresponding superfields of BF theory (cf. eqs. (4.2.6b), (4.2.6d)). Since S_{BV} solves the BV master equation (4.2.1), the BV superfield variation operator δ_{BV} satisfies (4.2.2), and thus is

nilpotent. For the same reason, the BV action S_{BV} satisfies (4.2.3) and so is BV invariant.

Let us now analyze the issue of gauge covariance in AKSZ $\mathfrak{v}$ CS gauge theory. As $\mathfrak{v}$ CS gauge theory involves in an essential way an invariant metric $(\cdot,\cdot)$, the relevant symmetry group of $\mathfrak{v}$ is not the general 1–automorphism group $\mathrm{Aut}_1(\mathfrak{v})$ (cf. subsect. 2.9) but its unitary subgroup $\mathrm{UAut}_1(\mathfrak{v})$. By definition, $\mathrm{UAut}_1(\mathfrak{v})$ consists of the 1–automorphisms $\phi \in \mathrm{Aut}_1(\mathfrak{v})$ such that $(\phi_0(x), \phi_0(y)) = (x, y)$ [20]. Correspondingly, only unitary gauge transformations are to be considered. These form the unitary subgroup $\mathrm{UGau}_1(M, \mathfrak{v})$ of $\mathrm{Gau}_1(M, \mathfrak{v})$.

For any unitary gauge transformation $g \in \mathrm{UGau}_1(N, \mathfrak{v})$, the gauge transformed basic superfields are

$$ {}^{g}\boldsymbol{q} = g_0(\boldsymbol{q} + \boldsymbol{\sigma}_g), \tag{4.6.10a} $$

$$ {}^{g}\boldsymbol{Q} = g_1(\boldsymbol{Q} - \boldsymbol{\Sigma}_g - \boldsymbol{\tau}_g(\boldsymbol{q} + \boldsymbol{\sigma}_g)) - \frac{1}{2} g_2(\boldsymbol{q} + \boldsymbol{\sigma}_g, \boldsymbol{q} + \boldsymbol{\sigma}_g). \tag{4.6.10b} $$

These expressions are identical in form to those of the corresponding gauge transformed superfields of BF theory (cf. eqs. (4.2.7b), (4.2.7d)).

Having defined the gauge transformation prescription, we can tackle the problem of the global definedness of AKSZ $\mathfrak{v}$ CS gauge theory. The BV form Ω_{BV} given in (4.6.7) turns out to be gauge invariant. Thus, it can be defined globally on a general 4–fold N. Conversely, as in ordinary CS theory, the BV action S_{BV} given in (4.6.8) is not gauge invariant. So, S_{BV} cannot be defined globally on a general N in the usual way. Nevertheless, there is an alternative way of giving a global meaning to S_{BV} proceeding as follows. Pick a background connection doublet $(\bar{\omega}, \bar{\Omega})$. Consider the superfield

$$ \boldsymbol{\Delta\mathcal{L}} = \boldsymbol{\mathcal{L}} - \bar{\boldsymbol{\mathcal{L}}} - \boldsymbol{d\Lambda}, \tag{4.6.11} $$

[20] It can be shown that, for $\phi = (\phi_0, \phi_1, \phi_2) \in \mathrm{UAut}_1(\mathfrak{v})$, $\phi_1 = \phi_0|_{\hat{\mathfrak{v}}_0}$ and $({\phi_1}^{-1}\phi_2(x,y), z)$ $+(y, {\phi_1}^{-1}\phi_2(x,z)) = 0$.

where $\boldsymbol{\mathcal{L}}$ is the integrand superfield in (4.6.8), $\bar{\boldsymbol{\mathcal{L}}}$ is $\boldsymbol{\mathcal{L}}$ with $(\boldsymbol{q}, \boldsymbol{Q}) = (-\bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\Omega}})$ and

$$\boldsymbol{\Lambda} = \frac{1}{2}(\bar{\boldsymbol{\omega}} + \boldsymbol{q}, \bar{\boldsymbol{\Omega}} + \boldsymbol{Q} - \frac{1}{6}[\bar{\boldsymbol{\omega}} + \boldsymbol{q}, \bar{\boldsymbol{\omega}} + \boldsymbol{q}]). \tag{4.6.12}$$

Then, thanks to Stokes' theorem, one has

$$S_{\mathrm{BV}} = \bar{S}_{\mathrm{BV}} + \int_{T[1]N} \varrho \boldsymbol{\Delta\mathcal{L}}, \tag{4.6.13}$$

where $\bar{S}_{\mathrm{BV}}$ is

$$\bar{S}_{\mathrm{BV}} = \int_{T[1]N} \varrho \bar{\boldsymbol{\mathcal{L}}}. \tag{4.6.14}$$

It is verified that ${}^g\boldsymbol{\Delta\mathcal{L}} = \boldsymbol{\Delta\mathcal{L}}$ for $g \in \mathrm{UGau}_1(N, \mathfrak{v})$. Therefore, the second term in the right hand side can be defined globally also on a general 4–fold N. $\bar{S}_{\mathrm{BV}} = \bar{S}_{\mathrm{cl}}$ is just the classical action evaluated in the background $(\bar{\omega}, \bar{\Omega})$. Again, it cannot be interpreted as an ordinary integral and must be defined by other means. However, as it is a mere background term, it does not affect the dynamics at the classical level and, perturbatively, also at the quantum one. In this way, using (4.6.13), in the weaker sense we have explained, S_{BV} can be globally defined on a general 4–fold N.

It is important to realize that the superfield $\boldsymbol{\Lambda}$ given in eq. (4.6.12) is not by itself gauge invariant. The exact term $\boldsymbol{d\Lambda}$ appearing in the right hand side of (4.6.11), so, cannot be dropped without spoiling the gauge invariance of $\boldsymbol{\Delta\mathcal{L}}$. For the same reason, it cannot eliminated upon integration on $T[1]N$ using Stokes' theorem on a general 4–fold N. Further, a generic variation $\delta\boldsymbol{\Lambda}$ of $\boldsymbol{\Lambda}$ with respect to the superfields $\boldsymbol{q}$, $\boldsymbol{Q}$ is also not gauge invariant. Hence, upon integration, $\boldsymbol{d\Lambda}$ does not yield a topological invariant when N is topologically non trivial.

The BV superfield variations (4.6.9) are gauge covariant and, so, are also defined globally on a general 4–fold N. This is obvious, since the BV superfield variations and the gauge transformation rules are formally identical to those of BF theory, which are gauge covariant.

The basic superfields $\boldsymbol{q}$, $\boldsymbol{Q}$ have a component expansion of the form

$$\boldsymbol{q} = c - \omega + \Omega^+ - C^+ + \Gamma^+, \tag{4.6.15a}$$

$$\boldsymbol{Q} = \Gamma - C + \Omega + \omega^+ - c^+, \tag{4.6.15b}$$

where the terms in the right hand side are written down in increasing order of form degree and decreasing order of ghost number degree with a conventional choice signs. It is possible to express the BV action and the BV superfield variations in component fields. Working with components, it is then possible to gauge fix the model.

4.7 Relation to other formulations

We review some of the results of ref. [45] and compare them with those obtained in the present paper. Closely related results were obtained also in ref. [46].

A graded smooth manifold X is a smooth space of the form $V[1]$, the 1 step grade shift of an ordinary graded smooth vector bundle $V = \bigoplus_i V_i \to M$ on a manifold M. The local coordinates x^i of X are just the local trivialization coordinates of $V[1]$ seen as Grassmann valued. The algebra $C^\infty(X)$ of smooth functions on X is defined as the graded commutative algebra $\mathrm{Fun}(V[1])$ of smooth fiberwise polynomial functions on the bundle $V[1]$. An ordinary manifold M can be seen as a graded manifold $X = V[1]$, where V is M viewed as a vector bundle over M of vanishing rank.

A graded vector field D on a graded manifold X is a graded derivation operator $D : C^\infty(X) \to C^\infty(X)$. In local coordinates x^i of X, D has thus the familiar expansion $D = D^i\partial/\partial x^i$. Graded vector fields form a graded Lie algebra $\mathrm{Vect}(X)$ under graded commutation.

A differential graded manifold is a pair (X, Δ), where X is a graded manifold and $\Delta \in \mathrm{Vect}(X)$ is a grade 1 vector field satisfying the nilpotency condition

$[\varDelta, \varDelta] = 2\varDelta \circ \varDelta = 0$. The graded vector bundle V underlying a differential graded manifold $(X, \varDelta)$ acquires a L_∞–algebroid structure, whose Chevalley–Eilenberg algebra bundle and differential operator are $\mathrm{CE}(V) \simeq C^\infty(X)$ and $\mathcal{Q}_{\mathrm{CE}(V)} \simeq \varDelta$, respectively, by a reason analog to that by which an L_∞ algebra structure on a graded vector space $\mathfrak{v}$ is codified in the Chevalley–Eilenberg algebra and differential $\mathrm{CE}(\mathfrak{v})$ and $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ (cf. subsect. 3.1).

A graded manifold X is characterized by the graded commutative algebra of differential forms $\Omega^*(X)$. By definition, if $X = V[1]$ with V a graded vector bundle, then $\Omega^*(X)$ is the graded commutative algebra $\mathrm{Fun}(T[1]V[1])$ of smooth fiberwise polynomial functions on the 1 step grade shifted tangent bundle $T[1]V[1]$ of $V[1]$ [21]. The form degree of $\Omega^*(X)$ is the fiberwise polynomial degree of $\mathrm{Fun}(T[1]V[1])$. The de Rham differential of X is the differential operator on $\mathrm{Fun}(T[1]V[1])$ locally given by $d_X = \xi^i \partial/\partial x^i$, where ξ^i are the fiber coordinates of $T[1]V[1]$. The contraction and Lie derivative operators of a graded vector field $D \in \mathrm{Vect}(X)$ on X are the differential operators on $\mathrm{Fun}(T[1]V[1])$ locally given by $i_D = D^i \partial/\partial \xi^i$ and $l_D = D^i \partial/\partial x^i + (-1)^D \xi^j \partial D^i/\partial x^j \partial/\partial \xi^i$. The graded version of the familiar Cartan relations holds. This supergeometric framework reduces to the standard one in the case where X is an ordinary manifold M.

If $(X, \varDelta)$ is a differential graded manifold, then $\Omega^*(X)$ comes equipped with the grade 1 nilpotent differential operator $l_\varDelta$. If V is the L_∞-algebroid underlying X, it is natural to define the Weil algebroid bundle and differential operator of V to be $\mathrm{W}(V) \simeq \Omega^*(X)$ and $\mathcal{Q}_{\mathrm{W}(V)} \simeq d_X + l_\varDelta$, since, by its construction, $\Omega^*(X)$ is locally an extension of $C^\infty(X)$ by grade shifted generators, d_X is the associated shift operator and $l_\varDelta$ is the extension of the operator $\varDelta$ from $C^\infty(X)$ to $\Omega^*(X)$ anticommuting with d_X, much as the Weil algebra $\mathrm{W}(\mathfrak{v})$ and differential $\mathcal{Q}_{\mathrm{W}(\mathfrak{v})}$ of

[21] Here and in the following, we refer to the fibers of the bundle $T[1]V[1]$. However, the notion of smoothness adopted in this context requires also polynomiality with respect to the fibers of the bundle $V[1]$.

an L_∞–algebra $\mathfrak{v}$ extend the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{v})$ and differential $\mathcal{Q}_{\mathrm{CE}(\mathfrak{v})}$ (cf. subsect. 3.1).

A grade n symplectic differential graded manifold is a differential graded manifold (X, Δ) equipped with a grade n non singular 2–form $\omega \in \Omega^2(X)$ such that $d_X\omega = 0$ and $l_\Delta\omega = 0$. In virtue of its properties, ω is also an element of $\mathrm{W}(V)$ satisfying $\mathcal{Q}_{\mathrm{W}(V)}\omega = 0$, where V is the graded vector bundle underlying X, that is a Weil cocycle. It can be shown that there is an element $\varpi \in \mathrm{CE}(V)$ and an element $\Lambda \in \mathrm{W}(V)$ with the property that $\mathcal{Q}_{\mathrm{CE}(V)}\varpi = 0$ and that $\mathcal{Q}_{\mathrm{W}(V)}\Lambda = \omega$ and $i^*\Lambda = \varpi$, where $i^* : \mathrm{W}(V) \to \mathrm{CE}(V)$ is the differential graded commutative algebra morphism corresponding to the natural projection $i^* : \Omega^*(X) \to \Omega^0(X) \simeq C^\infty(X)$. The Chevalley–Eilenberg cocycle ϖ is said to be in transgression with ω and Λ is called the transgression or Chern–Simons element of ω. Explicit expressions of ϖ and Λ are available. ϖ is given by

$$\varpi = \frac{1}{n+1}\omega_{ij}\,\mathrm{gr}(x^i)x^i\Delta^j, \tag{4.7.1}$$

while Λ is given by

$$\Lambda = \frac{1}{n}\omega_{ij}\,\mathrm{gr}(x^i)x^i\mathcal{Q}_{\mathrm{W}(V)}x^j - \varpi. \tag{4.7.2}$$

According to the authors of refs. [45], the triple of data $(\omega, \Lambda, \varpi)$ defines an AKSZ sigma like model with base space any closed $n+1$ dimensional world volume Σ and target space X. Field space can be identified with the set of differential graded commutative algebra morphisms $\varphi : \mathrm{W}(V) \to \Omega^*(\Sigma) \simeq \mathrm{Fun}(T[1]\Sigma)$. The AKSZ sigma model classical BV master action is given explicitly by

$$S_{\mathrm{BV}}(\varphi) = \int_{T[1]\Sigma} \varphi(\Lambda). \tag{4.7.3}$$

Now, we are going to show that the models studied in this section can be obtained by the above general procedure. (4.7.3) provides the complete superfield version of $S_{\mathrm{BV}}(\varphi)$. The authors of [45] concentrate however on the truncation of $S_{\mathrm{BV}}(\varphi)$

to the ghost number 0 sector of field space, which is just the classical action. This is fine but it is no longer sufficient when aiming to gauge fix the associated field theory, a necessary step in the path toward its quantization, as we did above.

Let us now show that the $\mathfrak{v}$ BF gauge theory model of subsect. 4.2 is a special case of the general construction of refs. [45]. Consider the graded manifold $X = V[1]$, where V is the delooping $b\mathfrak{V}$ of the graded vector space $\mathfrak{V} = \hat{\mathfrak{v}}_0[0] \oplus \hat{\mathfrak{v}}_0{}^\vee[0] \oplus \hat{\mathfrak{v}}_1[1] \oplus \hat{\mathfrak{v}}_1{}^\vee[-1]$, that is $\mathfrak{V}$ viewed as a vector bundle over the singleton manifold. Denote by q, p, Q, P the coordinates of X corresponding to the direct summands of $\mathfrak{V}$ in the given order. It is straightforwardly verified that the grade 1 vector field $\varDelta \in \mathrm{Vect}(X)$ of components

$$\varDelta^p = [q,p]^\vee - [Q,P]^\vee - \frac{1}{2}[q,q,P]^\vee, \tag{4.7.4a}$$

$$\varDelta^q = \frac{1}{2}[q,q] - \partial Q, \tag{4.7.4b}$$

$$\varDelta^P = [q,P]^\vee - \partial^\vee p, \tag{4.7.4c}$$

$$\varDelta^Q = [q,Q] - \frac{1}{6}[q,q,q] \tag{4.7.4d}$$

is nilpotent. Thus, $(X, \varDelta)$ is a differential graded manifold. X is endowed with a natural grade 2 symplectic form ς invariant under $\varDelta$, viz

$$\varsigma = -\langle dp, dq\rangle + \langle dP, dQ\rangle, \tag{4.7.5}$$

so that $(X, \varDelta)$ is a grade 2 symplectic differential graded manifold. Using (4.7.1), we can easily get the Chevalley–Eilenberg cocycle ϖ in transgression with ς,

$$\varpi = \langle p, -\frac{1}{2}[q,q] + \partial Q\rangle - \langle P, -[q,Q] + \frac{1}{6}[q,q,q]\rangle \tag{4.7.6}$$

and from this, using (4.7.2), the associated Chern–Simons element $\varLambda$,

$$\varLambda = -\langle p, \mathcal{Q}_{W(V)}q - \frac{1}{2}[q,q] + \partial Q\rangle + \langle P, \mathcal{Q}_{W(V)}Q - [q,Q] + \frac{1}{6}[q,q,q]\rangle, \tag{4.7.7}$$

up to an irrelevant $\mathcal{Q}_{W(V)}$–exact term. Application of (4.7.3) yields immediately the BV action of the BF theory given in (4.2.5). We conclude by noticing that,

as a byproduct, we have shown that there exists an L_∞ algebra $\mathfrak{V}$ canonically associated with any given 2–term L_∞ algebra $\mathfrak{v}$. As a graded vector space, $\mathfrak{V} = \mathfrak{v} \oplus \tilde{\mathfrak{v}}^\vee := \tilde{T}^*\mathfrak{v}$, where $\tilde{\mathfrak{v}}^\vee$ is the dual space of $\mathfrak{v}$ with sign reversed grading.

Our $\mathfrak{V}$ can be viewed in yet another way: as a certain Courant algebroid E. E is the trivial bundle $\hat{\mathfrak{v}}_1{}^\vee \times (\hat{\mathfrak{v}}_0 \oplus \hat{\mathfrak{v}}_0{}^\vee) \to \hat{\mathfrak{v}}_1{}^\vee$. The Courant bracket structure on E is that naturally yielded by the brackets $[\cdot,\cdot]$, $[\cdot,\cdot]^\vee$ on $\hat{\mathfrak{v}}_1{}^\vee \times (\hat{\mathfrak{v}}_0 \oplus \hat{\mathfrak{v}}_0{}^\vee)$. The anchor of E is induced by the bilinear form $\langle \cdot, \partial \cdot \rangle : \hat{\mathfrak{v}}_0{}^\vee \times \hat{\mathfrak{v}}_1 \to \mathbb{R}$ extended trivially to one $(\hat{\mathfrak{v}}_0 \oplus \hat{\mathfrak{v}}_0{}^\vee) \times \hat{\mathfrak{v}}_1 \to \mathbb{R}$ and then viewed as a mapping $\hat{\mathfrak{v}}_1{}^\vee \times (\hat{\mathfrak{v}}_0 \oplus \hat{\mathfrak{v}}_0{}^\vee) \to \hat{\mathfrak{v}}_1{}^\vee \times \hat{\mathfrak{v}}_1{}^\vee \simeq T\hat{\mathfrak{v}}_1{}^\vee$. The metric of E is just the fiberwise off diagonal symmetric bilinear form on $\hat{\mathfrak{v}}_0 \oplus \hat{\mathfrak{v}}_0{}^\vee$ determined by the duality pairing $\langle \cdot, \cdot \rangle : \hat{\mathfrak{v}}_0{}^\vee \times \hat{\mathfrak{v}}_0 \to \mathbb{R}$. Our BF theory is therefore a particular case of the Courant algebroid AKSZ sigma model first studied in ref. [70] and reexamined in ref. [45] in the above framework. In this paper, we went a step beyond those endeavours by obtaining the full superfield form of the BV action and carrying out the gauge fixing of the field theory.

Let us now show that the $\mathfrak{v}$ CS gauge theory model formulated in subsect. 4.6 also is a special case of the construction of refs. [45]. Consider the graded manifold $X = V[1]$, where V is the delooping $b\mathfrak{V}$ of the graded vector space $\mathfrak{V} = \hat{\mathfrak{v}}_0[0] \oplus \hat{\mathfrak{v}}_1[1]$. Denote by q, Q the coordinates of X corresponding to the direct summands of $\mathfrak{V}$ in the given order. It is readily checked that the grade 1 vector field $\varDelta \in \mathrm{Vect}(X)$ of components

$$\varDelta^q = \frac{1}{2}[q, q] - Q, \tag{4.7.8a}$$

$$\varDelta^Q = [q, Q] - \frac{1}{6}[q, q, q] \tag{4.7.8b}$$

is nilpotent. Thus, $(X, \varDelta)$ is a differential graded manifold. X is endowed with a natural grade 3 symplectic form ς invariant under $\varDelta$, viz

$$\varsigma = (dq, dQ), \tag{4.7.9}$$

so that $(X, \varDelta)$ is a grade 3 symplectic differential graded manifold. Using (4.7.1),

we can easily get the Chevalley–Eilenberg cocycle ϖ in transgression with ς,

$$\varpi = -(Q, -\frac{1}{2}[q,q] + Q) - \frac{1}{24}(q, [q,q,q]) \tag{4.7.10}$$

and from this, using (4.7.2), obtain the associated Chern–Simons element Λ,

$$\Lambda = (Q, \mathcal{Q}_{W(V)}q - \frac{1}{2}[q,q] + \frac{1}{2}\partial Q), + \frac{1}{24}(q, [q,q,q]). \tag{4.7.11}$$

again up to an irrelevant $\mathcal{Q}_{W(V)}$–exact term. Application of (4.7.3) yields immediately the BV action of the CS theory given in (4.6.8). Note that $\mathfrak{V}$ as an L_∞ algebra is just the 2–term L_∞ algebra $\mathfrak{v}$ we started with.

5 Conclusions and outlook

5.1 Summary of results

In this paper, we have worked out a version of semistrict higher gauge theory, whose symmetry is encoded in a semistrict Lie 2–algebra. This extends previous constructions which relied instead on differential Lie crossed modules [16–19].

In our formulation of semistrict higher gauge theory, the symmetry is encoded in a finite dimensional 2–term L_∞ algebra $\mathfrak{v}$ at infinitesimal level and in the automorphism 2–group $\mathrm{Aut}(\mathfrak{v})$ of $\mathfrak{v}$ at finite level. The basic datum is thus the algebra $\mathfrak{v}$. In this way, we avoid any reference to any Lie 2–group V integrating $\mathfrak{v}$, which may be infinite dimensional or may be something more general than a mere coherent 2–group, and rely instead on the 2–group $\mathrm{Aut}(\mathfrak{v})$, which is always finite dimensional and strict. Gauge transformations are mappings valued in the 1–cell group $\mathrm{Aut}_1(\mathfrak{v})$ of $\mathrm{Aut}(\mathfrak{v})$ together with a flat connection doublet and other form data satisfying a set of relations. Gauge transformations on a neighborhood O form an infinite dimensional group $\mathrm{Gau}_1(O,\mathfrak{v})$, which is the 1–cell group of a strict 2–group $\mathrm{Gau}(O,\mathfrak{v})$. A left action of $\mathrm{Gau}_1(O,\mathfrak{v})$ on fields on O is defined and gauge invariant Lagrangian field theoretic models can be built.

This approach has its advantages and disadvantages. At the differential level, it is very efficient and provides a powerful algorithm for the construction of local semistrict higher gauge models in perturbative Lagrangian field theory. At the integral level, it is not suitable for the study and efficient computation of higher parallel transport even in the strict theory and thus also for the investigation of non perturbative issues.

Using the BV quantization approach in the AKSZ geometrical version, we have been able to construct semistrict higher gauge theoretic generalizations of BF and Chern-Simons theory. These field theories are interesting as exemplification of our methodology and for the relation they bear with the general AKSZ construction

of refs. [45, 46]. Other field theoretic models with 2–term L_∞ algebra symmetry can conceivably be worked out. Relatedly, a number of issues call for further investigation. They are reviewed briefly below.

5.2 2–term L_∞ algebra Yang–Mills gauge theory

Though we have not investigated this in detail in this paper, it is not very difficult to construct a 2–term L_∞ algebra Yang–Mills gauge theory. Let $\mathfrak{v}$ be a 2–term L_∞ algebra and let X be any closed oriented manifold. We assume that $\mathfrak{v}$ is reduced and equipped with an invariant metric $(\cdot,\cdot)$ (cf. subsect. 4.6). We assume further that a gauge rectifier (λ,μ) has been chosen and that X is equipped with a Riemannian metric h. The basic fields of $\mathfrak{v}$ Yang–Mills gauge theory are the components of a connection doublet (ω,Ω) with curvature doublet (f,F). The action of the model is

$$S_{\rm YM}=\int_X\Big[\frac{1}{2}(\tilde f,*\tilde f)+\frac{1}{2}(\tilde F,*\tilde F)\Big], \tag{5.2.1}$$

where $(\tilde f,\tilde F)$ is the rectified curvature doublet

$$\tilde f=f, \tag{5.2.2a}$$

$$\tilde F=F+\lambda(\omega,f)-\mu(f). \tag{5.2.2b}$$

By construction, if the connection doublet (ω,Ω) is canonical, the action $S_{\rm YM}$ is gauge invariant and, so, globally definable. Though $\mathfrak{v}$ Yang–Mills gauge theory is a straightforward generalization of customary Yang–Mills gauge theory, the field equations derived from $S_{\rm YM}$ are rather more involved. They read

$$\begin{aligned}d(*\tilde f-\lambda^t(\omega,*\tilde F)+\mu^t(*\tilde F))+[\omega,*\tilde f-\lambda^t(\omega,*\tilde F)+\mu^t(*\tilde F)]&\\ +[\Omega,*\tilde F]+\frac{1}{2}[\omega,\omega,*\tilde F]+\lambda^t(f,*\tilde F)=0,&\end{aligned} \tag{5.2.3a}$$

$$d*\tilde F+[\omega,*\tilde F]+*\tilde f-\lambda^t(\omega,*\tilde F)+\mu^t(*\tilde F)=0, \tag{5.2.3b}$$

where (λ^t, μ^t) is the transposed rectifier of (λ, μ) with respect to the invariant metric and is defined in a way formally analogous to that of the dual rectifier $(\lambda^\vee, \mu^\vee)$. The complete symmetries of $\mathfrak{v}$ Yang–Mills gauge theory are not known to us. It remains to be seen whether the model has any physical applications.

5.3 Adding matter

Adding matter in 2–term L_∞ algebra gauge theory is a delicate issue. Suppose that the matter fields are valued in some linear space W and that the symmetry 2–term L_∞ algebra $\mathfrak{v}$ acts on matter fields linearly. Then, with every $x \in \mathfrak{v}_0$, there is associated an element $T_x \in \mathfrak{gl}(W)$ representing the action of x on those fields. It is reasonable to suppose that T is a representation, so that

$$[T_x, T_y] = T_{[x,y]}. \tag{5.3.1}$$

Since $\mathfrak{gl}(W)$ is an ordinary Lie algebra, the Jacobi identity holds in $\mathfrak{gl}(W)$. By (2.2.1c), this implies that $T_{\partial[x,y,z]} = 0$ identically. In general, so, we must have

$$T_{\partial X} = 0, \tag{5.3.2}$$

for every $X \in \mathfrak{v}_1$. By (2.2.1a), $\operatorname{im}\partial$ is an ideal of $\mathfrak{v}_0$ and, by (2.2.1c), $\mathfrak{g} = \mathfrak{v}_0/\operatorname{im}\partial$ is an ordinary Lie algebra. Therefore, a linear action of $\mathfrak{v}$ on matter fields reduces to one of the Lie algebra $\mathfrak{g}$. The non standard features of $\mathfrak{v}$ get in this way lost by the representation T. The natural question arises about whether other forms of linear action on matter fields can be defined, which faithfully reproduce the richer algebraic structure of $\mathfrak{v}$.

A AKSZ $\mathfrak{v}$ BF gauge theory in components

Substituting the component expansions (4.2.8) of the superfields into the expressions (4.2.5), (4.2.6) of the BV action $S_{\rm BV}$ and field variation $\delta_{\rm BV}$ of AKSZ $\mathfrak{v}$ BF gauge theory, we obtain the corresponding expressions in components.

The components organize in a number of doublets: the connection doublet $(\omega, \mathit{\Omega})$, its curvature doublet (f, F), four field doublets (b, B), (c, C), $(0, \mathit{\Gamma})$, $(\beta, 0)$ of bidegree $(2, -1)$, $(0, 1)$, $(-1, 2)$, $(3, -2)$ and five dual field doublets $(\mathit{\Omega}^+, \omega^+)$, (B^+, b^+), (C^+, c^+), $(\mathit{\Gamma}^+, 0)$, $(0, \beta^+)$, of bidegree $(1, -1)$, $(0, 0)$, $(2, -2)$, $(3, -3)$, $(-1, 1)$, respectively, and their covariant derivative doublets (cf. subsect. 3.2).

The component expansion of the BV action $S_{\rm BV}$ is

$$\begin{aligned}S_{\rm BV} = \int_M \Big[&\langle b^+, f + [c, b]\rangle - \langle B^+, F + [c, B] + [b, C] + [\beta, \mathit{\Gamma}] - [\omega, c, b] \\ &- \frac{1}{2}[c, c, \beta]\rangle + \langle \omega^+, Dc\rangle - \langle \mathit{\Omega}^+, DC + [b, \mathit{\Gamma}] - \frac{1}{2}[c, c, b]\rangle \\ &+ \langle c^+, \frac{1}{2}[c, c] - \partial \mathit{\Gamma}\rangle - \langle C^+, D\mathit{\Gamma} + [c, C] - \frac{1}{2}[\omega, c, c]\rangle \\ &+ \langle \beta^+, Db + [c, \beta]\rangle - \langle \mathit{\Gamma}^+, [c, \mathit{\Gamma}] - \frac{1}{6}[c, c, c]\rangle\Big]\end{aligned} \tag{A.0.3}$$

The component expansion of the BV field variation $\delta_{\rm BV}$ reads

$$\delta_{\rm BV}\beta^+ = -[c, \beta^+]^\vee + [\mathit{\Gamma}, B^+]^\vee + \frac{1}{2}[c, c, B^+]^\vee, \tag{A.0.4a}$$

$$\begin{aligned}\delta_{\rm BV}b^+ = -D\beta^+ - [c, b^+]^\vee + [\mathit{\Gamma}, \mathit{\Omega}^+]^\vee + [C, B^+]^\vee \\ + [\omega, c, B^+]^\vee + \frac{1}{2}[c, c, \mathit{\Omega}^+]^\vee,\end{aligned} \tag{A.0.4b}$$

$$\begin{aligned}\delta_{\rm BV}\omega^+ = -Db^+ - [c, \omega^+]^\vee - [b, \beta^+]^\vee + [C, \mathit{\Omega}^+]^\vee + [\mathit{\Gamma}, C^+]^\vee \\ + [c, b, B^+]^\vee + +[\omega, c, \mathit{\Omega}^+]^\vee, +\frac{1}{2}[c, c, C^+]^\vee,\end{aligned} \tag{A.0.4c}$$

$$\begin{aligned}\delta_{\rm BV}c^+ = -D\omega^+ - [c, c^+]^\vee - [b, b^+]^\vee - [\beta, \beta^+]^\vee + [\mathit{\Gamma}, \mathit{\Gamma}^+]^\vee \\ + [C, C^+]^\vee + [B, B^+]^\vee + [c, \beta, B^+]^\vee + [\omega, b, B^+]^\vee \\ + [c, b, \mathit{\Omega}^+]^\vee + [\omega, c, C^+]^\vee + \frac{1}{2}[c, c, \mathit{\Gamma}^+]^\vee,\end{aligned} \tag{A.0.4d}$$

$$\delta_{\mathrm{BV}} c = -\frac{1}{2}[c, c] + \partial \varGamma, \tag{A.0.4e}$$

$$\delta_{\mathrm{BV}} \omega = -Dc, \tag{A.0.4f}$$

$$\delta_{\mathrm{BV}} b = -f - [c, b], \tag{A.0.4g}$$

$$\delta_{\mathrm{BV}} \beta = -Db - [c, \beta], \tag{A.0.4h}$$

$$\delta_{\mathrm{BV}} B^+ = -[c, B^+]^\vee + \partial^\vee \beta^+, \tag{A.0.4i}$$

$$\delta_{\mathrm{BV}} \varOmega^+ = -DB^+ - [c, \varOmega^+]^\vee, \tag{A.0.4j}$$

$$\delta_{\mathrm{BV}} C^+ = -D\varOmega^+ - [c, C^+]^\vee - [b, B^+]^\vee, \tag{A.0.4k}$$

$$\delta_{\mathrm{BV}} \varGamma^+ = -DC^+ - [c, \varGamma^+]^\vee - [b, \varOmega^+]^\vee - [\beta, B^+]^\vee, \tag{A.0.4l}$$

$$\delta_{\mathrm{BV}} \varGamma = -[c, \varGamma] + \frac{1}{6}[c, c, c], \tag{A.0.4m}$$

$$\delta_{\mathrm{BV}} C = -D\varGamma - [c, C] + \frac{1}{2}[\omega, c, c], \tag{A.0.4n}$$

$$\delta_{\mathrm{BV}} \varOmega = -DC - [b, \varGamma] + \frac{1}{2}[c, c, b], \tag{A.0.4o}$$

$$\delta_{\mathrm{BV}} B = -F - [c, B] - [b, C] - [\beta, \varGamma] + [\omega, c, b] + \frac{1}{2}[c, c, \beta]. \tag{A.0.4p}$$

Under a gauge transformation $g \in \mathrm{Gau}(M, \mathfrak{v})$, the component fields transform as follows

$${}^g\beta^+ = {g^\vee}_0(\beta^+) - {g^\vee}_2(g_0(c), B^+), \tag{A.0.5a}$$

$$\begin{aligned} {}^g b^+ = {g^\vee}_0(b^+ - {\tau_g}^\vee(B^+)) - {g^\vee}_2(g_0(\omega - \sigma_g), B^+) \\ - {g^\vee}_2(g_0(c), \varOmega^+), \end{aligned} \tag{A.0.5b}$$

$$\begin{aligned} {}^g\omega^+ = {g^\vee}_0(\omega^+ - {\tau_g}^\vee(\varOmega^+)) - {g^\vee}_2(g_0(b), B^+) \\ - {g^\vee}_2(g_0(\omega - \sigma_g), \varOmega^+) - {g^\vee}_2(g_0(c), C^+), \end{aligned} \tag{A.0.5c}$$

$$\begin{aligned} {}^g c^+ = {g^\vee}_0(c^+ - {\tau_g}^\vee(C^+)) - {g^\vee}_2(g_0(\beta), B^+) \\ - {g^\vee}_2(g_0(b), \varOmega^+) - {g^\vee}_2(g_0(\omega - \sigma_g), C^+) - {g^\vee}_2(g_0(c), \varGamma^+), \end{aligned} \tag{A.0.5d}$$

$${}^g c = g_0(c), \tag{A.0.5e}$$

$$ {}^{g}\omega = g_0(\omega - \sigma_g), \tag{A.0.5f} $$

$$ {}^{g}b = g_0(b), \tag{A.0.5g} $$

$$ {}^{g}\beta = g_0(\beta), \tag{A.0.5h} $$

$$ {}^{g}B^{+} = g^{\vee}{}_{1}(B^{+}) \tag{A.0.5i} $$

$$ {}^{g}\Omega^{+} = g^{\vee}{}_{1}(\Omega^{+}) \tag{A.0.5j} $$

$$ {}^{g}C^{+} = g^{\vee}{}_{1}(C^{+}) \tag{A.0.5k} $$

$$ {}^{g}\Gamma^{+} = g^{\vee}{}_{1}(\Gamma^{+}) \tag{A.0.5l} $$

$$ {}^{g}\Gamma = g_1(\Gamma) - \frac{1}{2}g_2(c, c), \tag{A.0.5m} $$

$$ {}^{g}C = g_1(C + \tau_g(c)) - g_2(\omega - \sigma_g, c), \tag{A.0.5n} $$

$$ {}^{g}\Omega = g_1(\Omega - \Sigma_g + \tau_g(\omega - \sigma_g)) - \frac{1}{2}g_2(\omega - \sigma_g, \omega - \sigma_g) - g_2(b, c), \tag{A.0.5o} $$

$$ {}^{g}B = g_1(B + \tau_g(b)) - g_2(\omega - \sigma_g, b) - g_2(c, \beta). \tag{A.0.5p} $$

B Rectified AKSZ $\mathfrak{v}$ BF gauge theory in components

Substituting the component expansions (4.2.8) of the superfields into the expressions (4.3.1), (4.3.2) of the BV action $S_{\rm BV}$ and field variation $\delta_{\rm BV}$ of rectified AKSZ $\mathfrak{v}$ BF gauge theory, we obtain the corresponding expressions in components. The components organize in a number of doublets in the same way as in the non rectified theory described in app. A.

The component expansion of the BV action $S_{\rm BV}$ reads

$$\begin{aligned}
S_{\rm BV} = \int_M \Big[&\langle b^+, \bar{f}_{\lambda,\mu} + \bar{D}_{\lambda,\mu}\omega + \frac{1}{2}[\omega,\omega]_\lambda + [c,b]_\lambda - \partial\Omega\rangle \\
&- \langle B^+, \bar{F}_{\lambda,\mu} + \bar{D}_{\lambda,\mu}\Omega + [\omega, \Omega + \bar{\Omega} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda \\
&+ [c,B]_\lambda + [b,C]_\lambda + [\beta,\Gamma]_\lambda - [\omega+\bar{\omega},c,b]_\lambda - \frac{1}{2}[c,c,\beta]_\lambda \\
&- \frac{1}{6}[\omega+\bar{\omega},\omega+\bar{\omega},\omega+\bar{\omega}]_\lambda + \frac{1}{6}[\bar{\omega},\bar{\omega},\bar{\omega}]_\lambda + v_{\lambda,\mu}(c,b) \\
&+ \frac{1}{2}v_{\lambda,\mu}(\omega+\bar{\omega},\omega+\bar{\omega}) - \frac{1}{2}v_{\lambda,\mu}(\bar{\omega},\bar{\omega}) - w_{\lambda,\mu}(\omega)\rangle \\
&+ \langle \omega^+, \bar{D}_{\lambda,\mu}c + [\omega,c]_\lambda - \partial C\rangle \\
&- \langle \Omega^+, \bar{D}_{\lambda,\mu}C + [\omega,C]_\lambda + [c, \Omega + \bar{\Omega} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda \\
&+ [b,\Gamma]_\lambda - \frac{1}{2}[c,c,b]_\lambda - \frac{1}{2}[\omega+\bar{\omega},\omega+\bar{\omega},c]_\lambda \\
&+ v_{\lambda,\mu}(\omega+\bar{\omega},c) - w_{\lambda,\mu}(c)\rangle + \langle c^+, \frac{1}{2}[c,c]_\lambda - \partial\Gamma\rangle \\
&- \langle C^+, \bar{D}_{\lambda,\mu}\Gamma + [\omega,\Gamma]_\lambda + [c,C]_\lambda - \frac{1}{2}[\omega+\bar{\omega},c,c]_\lambda \\
&+ \frac{1}{2}v_{\lambda,\mu}(c,c)\rangle + \langle \beta^+, \bar{D}_{\lambda,\mu}b + [\omega,b]_\lambda + [c,\beta]_\lambda - \partial B\rangle \\
&- \langle \Gamma^+, [c,\Gamma]_\lambda - \frac{1}{6}[c,c,c]_\lambda\rangle\Big]_\lambda
\end{aligned} \tag{B.0.6}$$

The component expansion of the BV field variation $\delta_{\rm BV}$ is

$$\delta_{\rm BV}\beta^+ = -[c,\beta^+]_\lambda{}^\vee + [\Gamma,B^+]_\lambda{}^\vee + \frac{1}{2}[c,c,B^+]_\lambda{}^\vee, \tag{B.0.7a}$$

$$\delta_{\rm BV}b^+ = -\bar{D}_{\lambda,\mu}\beta^+ - [\omega,\beta^+]_\lambda{}^\vee - [c,b^+]_\lambda{}^\vee + [\Gamma,\Omega^+]_\lambda{}^\vee \tag{B.0.7b}$$

$$+ [C, B^+]_\lambda{}^\vee + [\omega+\bar{\omega}, c, B^+]_\lambda{}^\vee + \frac{1}{2}[c, c, \mathit{\Omega}^+]_\lambda{}^\vee - v_{\lambda,\mu}{}^\vee(c, B^+),$$

$$\delta_{\mathrm{BV}}\omega^+ = -\bar{D}_{\lambda,\mu}b^+ - [\omega, b^+]_\lambda{}^\vee - [c, \omega^+]_\lambda{}^\vee - [b, \beta^+]_\lambda{}^\vee \tag{B.0.7c}$$

$$+ [\mathit{\Omega}+\bar{\mathit{\Omega}} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega}), B^+]_\lambda{}^\vee + [\mathit{\Gamma}, C^+]_\lambda{}^\vee + [C, \mathit{\Omega}^+]_\lambda{}^\vee$$

$$+ \frac{1}{2}[\omega+\bar{\omega}, \omega+\bar{\omega}, B^+]_\lambda{}^\vee + [c, b, B^+]_\lambda{}^\vee + [\omega+\bar{\omega}, c, \mathit{\Omega}^+]_\lambda{}^\vee$$

$$+ \frac{1}{2}[c, c, C^+]_\lambda{}^\vee - v_{\lambda,\mu}{}^\vee(\omega+\bar{\omega}, B^+) - v_{\lambda,\mu}{}^\vee(c, \mathit{\Omega}^+) + w_{\lambda,\mu}{}^\vee(B^+),$$

$$\delta_{\mathrm{BV}}c^+ = -\bar{D}_{\lambda,\mu}\omega^+ - [\omega, \omega^+]_\lambda{}^\vee - [c, c^+]_\lambda{}^\vee - [b, b^+]_\lambda{}^\vee - [\beta, \beta^+]_\lambda{}^\vee \tag{B.0.7d}$$

$$+ [\mathit{\Omega}+\bar{\mathit{\Omega}} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega}), \mathit{\Omega}^+]_\lambda{}^\vee + [\mathit{\Gamma}, \mathit{\Gamma}^+]_\lambda{}^\vee + [C, C^+]_\lambda{}^\vee$$

$$+ [B, B^+]_\lambda{}^\vee + [c, \beta, B^+]_\lambda{}^\vee + [\omega+\bar{\omega}, b, B^+]_\lambda{}^\vee + [c, b, \mathit{\Omega}^+]_\lambda{}^\vee$$

$$+ \frac{1}{2}[\omega+\bar{\omega}, \omega+\bar{\omega}, \mathit{\Omega}^+]_\lambda{}^\vee + [\omega+\bar{\omega}, c, C^+]_\lambda{}^\vee + \frac{1}{2}[c, c, \mathit{\Gamma}^+]_\lambda{}^\vee,$$

$$- v_{\lambda,\mu}{}^\vee(b, B^+) - v_{\lambda,\mu}{}^\vee(\omega+\bar{\omega}, \mathit{\Omega}^+) - v_{\lambda,\mu}{}^\vee(c, C^+) + w_{\lambda,\mu}{}^\vee(\mathit{\Omega}^+),$$

$$\delta_{\mathrm{BV}}c = -\frac{1}{2}[c, c]_\lambda + \partial\mathit{\Gamma}, \tag{B.0.7e}$$

$$\delta_{\mathrm{BV}}\omega = -\bar{D}_{\lambda,\mu}c - [\omega, c]_\lambda + \partial C, \tag{B.0.7f}$$

$$\delta_{\mathrm{BV}}b = -\bar{f}_{\lambda,\mu} - \bar{D}_{\lambda,\mu}\omega - \frac{1}{2}[\omega, \omega]_\lambda - [c, b]_\lambda + \partial\mathit{\Omega}, \tag{B.0.7g}$$

$$\delta_{\mathrm{BV}}\beta = -\bar{D}_{\lambda,\mu}b - [\omega, b]_\lambda - [c, \beta]_\lambda + \partial B, \tag{B.0.7h}$$

$$\delta_{\mathrm{BV}}B^+ = -[c, B^+]_\lambda{}^\vee + \partial^\vee\beta^+, \tag{B.0.7i}$$

$$\delta_{\mathrm{BV}}\mathit{\Omega}^+ = -\bar{D}_{\lambda,\mu}B^+ - [\omega, B^+]_\lambda{}^\vee - [c, \mathit{\Omega}^+]_\lambda{}^\vee + \partial^\vee b^+, \tag{B.0.7j}$$

$$\delta_{\mathrm{BV}}C^+ = -\bar{D}_{\lambda,\mu}\mathit{\Omega}^+ - [\omega, \mathit{\Omega}^+]_\lambda{}^\vee - [c, C^+]_\lambda{}^\vee - [b, B^+]_\lambda{}^\vee + \partial^\vee\omega^+, \tag{B.0.7k}$$

$$\delta_{\mathrm{BV}}\mathit{\Gamma}^+ = -\bar{D}_{\lambda,\mu}C^+ - [\omega, C^+]_\lambda{}^\vee - [c, \mathit{\Gamma}^+]_\lambda{}^\vee \tag{B.0.7l}$$

$$- [b, \mathit{\Omega}^+]_\lambda{}^\vee - [\beta, B^+]_\lambda{}^\vee + \partial^\vee c^+,$$

$$\delta_{\mathrm{BV}}\mathit{\Gamma} = -[c, \mathit{\Gamma}]_\lambda + \frac{1}{6}[c, c, c]_\lambda, \tag{B.0.7m}$$

$$\delta_{\mathrm{BV}}C = -\bar{D}_{\lambda,\mu}\mathit{\Gamma} - [\omega, \mathit{\Gamma}]_\lambda - [c, C]_\lambda + \frac{1}{2}[\omega+\bar{\omega}, c, c]_\lambda - \frac{1}{2}v_{\lambda,\mu}(c, c), \tag{B.0.7n}$$

$$\delta_{\mathrm{BV}}\mathit{\Omega} = -\bar{D}_{\lambda,\mu}C - [\omega, C]_\lambda - [c, \mathit{\Omega}+\bar{\mathit{\Omega}} - \frac{1}{2}\lambda(\bar{\omega},\bar{\omega}) + \mu(\bar{\omega})]_\lambda \tag{B.0.7o}$$

$$
\begin{aligned}
&- [b, \mathit{\Gamma}]_\lambda + \frac{1}{2}[c, c, b]_\lambda + \frac{1}{2}[\omega + \bar{\omega}, \omega + \bar{\omega}, c]_\lambda \\
&- v_{\lambda,\mu}(\omega + \bar{\omega}, c) + w_{\lambda,\mu}(c),
\end{aligned}
$$

$$
\begin{aligned}
\delta_{\mathrm{BV}} B = -\bar{F}_{\lambda,\mu} - \bar{D}_{\lambda,\mu}\mathit{\Omega} - [\omega, \mathit{\Omega} + \bar{\mathit{\Omega}} - \frac{1}{2}\lambda(\bar{\omega}, \bar{\omega}) + \mu(\bar{\omega})]_\lambda \qquad \text{(B.0.7p)} \\
- [c, B]_\lambda - [b, C]_\lambda - [\beta, \mathit{\Gamma}]_\lambda + [\omega + \bar{\omega}, c, b]_\lambda + \frac{1}{2}[c, c, \beta]_\lambda \\
+ \frac{1}{6}[\omega + \bar{\omega}, \omega + \bar{\omega}, \omega + \bar{\omega}]_\lambda - \frac{1}{6}[\bar{\omega}, \bar{\omega}, \bar{\omega}]_\lambda - v_{\lambda,\mu}(c, b) \\
- \frac{1}{2}v_{\lambda,\mu}(\omega + \bar{\omega}, \omega + \bar{\omega}) + \frac{1}{2}v_{\lambda,\mu}(\bar{\omega}, \bar{\omega}) + w_{\lambda,\mu}(\omega).
\end{aligned}
$$

Under a gauge transformation $g \in \mathrm{Gau}(M, \mathfrak{v})$, the component fields transform in an obvious way, since they are all rectified fields (cf. subsect. 3.5).

C The Lagrangian submanifold $\mathcal{L}$

The Lagrangian submanifold $\mathcal{L}$ of the BV field space $\mathcal{F}$ generated by the gauge fermion Ψ given by (4.4.7) is specified by the constraints

$$\beta^{+} = - * \tilde{q}, \tag{C.0.8a}$$

$$b^{+} = * \bar{D}_{\lambda,\mu} \tilde{\omega}, \tag{C.0.8b}$$

$$\omega^{+} = - * \bar{D}_{\lambda,\mu} \tilde{c}, \tag{C.0.8c}$$

$$c^{+} = 0, \tag{C.0.8d}$$

$$B^{+} = * \tilde{F}, \tag{C.0.8e}$$

$$\Omega^{+} = * \bar{D}_{\lambda,\mu} \tilde{C}, \tag{C.0.8f}$$

$$C^{+} = - * \bar{D}_{\lambda,\mu} \tilde{\Gamma}, \tag{C.0.8g}$$

$$\Gamma^{+} = 0, \tag{C.0.8h}$$

$$\tilde{c}^{+} = - \bar{D}_{\lambda,\mu} * \omega \tag{C.0.8i}$$

$$\tilde{\theta}^{+} = 0, \tag{C.0.8j}$$

$$\tilde{\omega}^{+} = - \bar{D}_{\lambda,\mu} * b - * \bar{D}_{\lambda,\mu} a, \tag{C.0.8k}$$

$$\tilde{\xi}^{+} = 0, \tag{C.0.8l}$$

$$\tilde{q}^{+} = * \beta \tag{C.0.8m}$$

$$\tilde{\chi}^{+} = 0, \tag{C.0.8n}$$

$$\tilde{\Gamma}^{+} = \bar{D}_{\lambda,\mu} * C, \tag{C.0.8o}$$

$$\tilde{H}^{+} = 0, \tag{C.0.8p}$$

$$\tilde{C}^{+} = \bar{D}_{\lambda,\mu} * \Omega - * \bar{D}_{\lambda,\mu} \Phi, \tag{C.0.8q}$$

$$\tilde{\Theta}^{+} = 0, \tag{C.0.8r}$$

$$\tilde{F}^{+} = - * B, \tag{C.0.8s}$$

$$\tilde{\Sigma}^{+} = 0 \tag{C.0.8t}$$

$$a^{+} = -\bar{D}_{\lambda,\mu} * \tilde{\omega} \tag{C.0.8u}$$

$$\theta^{+} = 0, \tag{C.0.8v}$$

$$\Phi^{+} = \bar{D}_{\lambda,\mu} * \tilde{C}, \tag{C.0.8w}$$

$$H^{+} = 0. \tag{C.0.8x}$$

Acknowledgement We thank the referee of the paper and C. Wockel for pointing out the mistakes and weaknesses of the first version of the paper and for providing useful suggestions for its improvement. The paper is dedicated to the memory of my mother, Lucia Baviera, who died while it was being completed.

References

[1] J. Polchinski, "String theory. Vol. 2: superstring theory and beyond", *Cambridge, UK: Univ. Pr. (1998) 531 p.*

[2] K. Becker, M. Becker and J. H. Schwarz, "String theory and M-theory: a modern introduction", *Cambridge, UK: Univ. Pr. (2007) 739 p.*

[3] C. V. Johnson, "D-branes", available at *Cambridge, USA: Univ. Pr. (2003) 548 p.*

[4] J. C. Baez, "An introduction to spin foam models of BF theory and quantum gravity", Lect. Notes Phys. **543** (2000) 25 [arXiv:gr-qc/9905087].

[5] C. Rovelli, "Quantum gravity", *Cambridge, UK: Univ. Pr. (2004) 455 p.*

[6] R. Savit, "Topological excitations in U(1) invariant theories", Phys. Rev. Lett. **39** (1977) 55.

[7] D. Z. Freedman and P. K. Townsend, "Antisymmetric tensor gauge theories and non linear sigma models", Nucl. Phys. B **177** (1981) 282.

[8] R. I. Nepomechie, "Approaches to a non Abelian antisymmetric tensor gauge field theory", Nucl. Phys. B **212** (1983) 301.

[9] C. Teitelboim, "Gauge Invariance for Extended Objects", Phys. Lett. B **167** (1986) 63.

[10] M. Henneaux and C. Teitelboim, "p form electrodynamics", Found. Phys. **16** (1986) 593.

[11] M. Blau and G. Thompson, "Topological gauge theories of antisymmetric tensor fields", Annals Phys. **205** (1991) 130.

[12] S. N. Solodukhin, "Non Abelian gauge antisymmetric tensor fields", Nuovo Cim. B **108** (1993) 1275 [arXiv:hep-th/9211046].

[13] A. Lahiri, "Gauge transformations of the non Abelian two form", Mod. Phys. Lett. A **17** (2002) 1643 [arXiv:hep-th/0107104].

[14] A. Lahiri, "Local symmetries of the non Abelian two form", J. Phys. A **35** (2002) 8779 [arXiv:hep-th/0109220].

[15] I. Chepelev, "Non Abelian Wilson surfaces", JHEP **0202** (2002) 013 [arXiv:hep-th/0111018].

[16] J. C. Baez, "Higher Yang-Mills theory", arXiv:hep-th/0206130.

[17] C. Hofman, "Non Abelian 2–forms", arXiv:hep-th/0207017.

[18] H. Pfeiffer, "Higher gauge theory and a non Abelian generalization of 2–form electrodynamics", Annals Phys. **308** (2003) 447 [arXiv:hep-th/0304074].

[19] J. C. Baez and J. Huerta, "An invitation to higher gauge theory", arXiv: 1003.4485 [hep-th].

[20] J. Baez and A. Lauda, "Higher dimensional algebra V: 2-groups", Theor. Appl. Categor. **12** (2004) 423 [arXiv:math.0307200].

[21] J. C. Baez and A. S. Crans, "Higher dimensional algebra VI: Lie 2–algebras", Theor. Appl. Categor. **12** (2004) 492 [arXiv:math/0307263].

[22] T. Lada and J. Stasheff, "Introduction to SH Lie algebras for physicists", Int. J. Theor. Phys. **32** (1993) 1087 [arXiv:hep-th/9209099].

[23] T. Lada and M. Markl, "Strongly homotopy Lie algebras", Comm. Algebra **23** (1995) 2147 [arXiv:hep-th/9406095].

[24] J. L. Brylinski, “Loop spaces, characteristic classes and geometric quantization”, *Boston, USA: Birkhaeuser (1993) (Progress in mathematics, 107) 300 p.*

[25] L. Breen and W. Messing, “Differential geometry of gerbes”, Adv. Math. **198** (2005) 732 [arXiv:math/0106083].

[26] A. S. Schwartz, “The partition function of degenerate quadratic functionals and Ray-Singer invariants”, Lett. Math. Phys. **2** (1978) 247.

[27] G. T. Horowitz, “Exactly soluble diffeomorphism invariant theories”, Commun. Math. Phys. **125** (1989) 417.

[28] H. Y. Guo, Y. Ling, R. S. Tung and Y. Z. Zhang, “Chern-Simons term for BF theory and gravity as a generalized topological field theory in four-dimensions”, Phys. Rev. D **66** (2002) 064017 [arXiv:hep-th/0204059].

[29] L. Freidel and A. Starodubtsev, “Quantum gravity in terms of topological observables”, arXiv:hep-th/0501191.

[30] F. Girelli and H. Pfeiffer, “Higher gauge theory: differential versus integral formulation”, J. Math. Phys. **45** (2004) 3949 [arXiv:hep-th/0309173].

[31] F. Girelli, H. Pfeiffer and E. M. Popescu, “Topological higher gauge theory - from BF to BFCG theory”, J. Math. Phys. **49** (2008) 032503 [arXiv:0708.3051 [hep-th]].

[32] J. Baez and U. Schreiber, “Higher gauge theory: 2-connections on 2-bundles”, arXiv:hep-th/0412325.

[33] J. C. Baez and U. Schreiber, “Higher gauge theory”, in *Categories in Algebra, Geometry and Mathematical Physics*, eds. A. Davydov et al., Contemp. Math. **431** AMS, Providence (USA) (2007) 7 [arXiv:math/0511710].

[34] P. Aschieri, L. Cantini and B. Jurco, “NonAbelian bundle gerbes, their differential geometry and gauge theory”, Commun. Math. Phys. **254** (2005) 367 [hep-th/0312154].

[35] D. Fiorenza, U. Schreiber and J. Stasheff “Čech cocycles for differential characteristic classes An ∞–Lie theoretic construction”, arXiv:1011.4735 [math.AT].

[36] U. Schreiber, “Differential cohomology in a cohesive ∞–topos”, available at http://ncatlab.org/schreiber/files/cohesivedocumentv032.pdf and references therein.

[37] I. A. Batalin and G. A. Vilkovisky, “Gauge algebra and quantization”, Phys. Lett. B **102** (1981) 27.

[38] I. A. Batalin and G. A. Vilkovisky, “Quantization of gauge theories with linearly dependent generators”, Phys. Rev. D **28** (1983) 2567 (Erratum-ibid. D **30** (1984) 508).

[39] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, “The geometry of the master equation and topological quantum field theory”, Int. J. Mod. Phys. A **12** (1997) 1405 [arXiv:hep-th/9502010].

[40] N. Ikeda, “Lectures on AKSZ Topological Field Theories for Physicists”, arXiv:1204.3714 [hep-th].

[41] R. Zucchini, “The Hitchin Model, Poisson-quasi-Nijenhuis Geometry and Symmetry Reduction”, JHEP **0710** (2007) 075 [arXiv:0706.1289 [hep-th]].

[42] R. Zucchini, “Gauging the Poisson sigma model,” JHEP **0805** (2008) 018 [arXiv:0801.0655 [hep-th]].

[43] E. Witten, “Topological Quantum Field Theory”, Commun. Math. Phys. **117** (1988) 353.

[44] E. Witten, “Quantum field theory and the Jones polynomial”, Commun. Math. Phys. **121** (1989) 351.

[45] D. Fiorenza, C. L. Rogers and U. Schreiber, “A Higher Chern-Weil derivation of AKSZ σ-models”, arXiv:1108.4378 [math-ph].

[46] A. Kotov and T. Strobl, “Characteristic classes associated to Q-bundles”, arXiv:0711.4106 [math.DG].

[47] M. Schlessinger and J. D. Stasheff, “The Lie algebra structure of tangent cohomology and deformation theory”, J. of Pure and Appl. Algebra **38** (1985) 313.

[48] M. Gerstenhaber, “A uniform cohomology theory for algebras”, Proc. Nat. Acad. Sci. **51** (1964) 626.

[49] I. Ya. Dorfman, “Dirac structures of integrable evolution equations”, Phys. Lett. A **125** (1987) 240.

[50] T. Courant, “Dirac manifolds”, Trans. Amer. Math. Soc. **319** (1990) 631.

[51] D. Roytenberg and A. Weinstein, “Courant algebroids and strongly homotopy Lie algebras”, Lett. Math. Phys. **46** (1998) 81 [arXiv:math/9802118].

[52] J. Baez, A. Crans, U. Schreiber and D. Stevenson, “From loop groups to 2-groups”, HHA **9** (2007), 101. [arXiv:math.QA/0504123].

[53] S. S. Chern and J. Simons, “Characteristic forms and geometric invariants”, Ann. Math. **99** (1974) 48.

[54] J. Cheeger and J. Simons, "Differential characters and geometric invariants", in *Geometry and Topology*, eds. J. Alexander and J. Harer, Lecture Notes in Mathematics **1167** (1985) 50.

[55] J. H. C. Whitehead, "Note on a previous paper entitled On adding relations to homotopy groups' ', Ann. Math. **47** (1946) 80

[56] R. Brown and C. B. Spencer, "G-groupoids, crossed modules, and the classifying space of a topological group", Proc. Kon. Akad. v. Wet. **79** (1976) 296.

[57] R. Moufang, "Zur Struktur von Alternativkörpern", Math. Ann. **110** (1935) 416.

[58] A. Pressley and G. Segal, "Loop groups" *Oxford U. K., Oxford University Press (1986).*

[59] A. Henriques, "Integrating L_∞ algebras", arXiv:math/0603563 .

[60] C. Schommer–Pries, "A finite dimensional string 2–group", arXiv:0911.2483.

[61] C. Chevalley and S. Eilenberg, "Cohomology theory of Lie groups and Lie algebras", Trans. Amer. Math. Soc. **63** (1948) 85.

[62] A. Weil, "Géométrie différentielle des espaces fibrés", *Letters to Chevalley and Koszul* (1949).

[63] H. Cartan, "Cohomologie reélle dun espace fibré principal différentielle I", *Sminaire Henri Cartan 1949/50*, CBRM (1950) 1901.

[64] H. Cartan, "Cohomologie reélle dun espace fibré principal différentielle II", *Sminaire Henri Cartan 1949/50*, CBRM (1950) 20-01.

[65] C. Ehresmann, “ Les connexions infinitésimales dans un espace fibré différentiable”, *Colloque de topologie (espaces fibrés) Bruxelles 1950*, Georges Thone, Liège, (1951) 29.

[66] T. Nikolaus and K. Waldorf, “Four equivalent versions of non-abelian gerbes”, arXiv:1103.4815 [math.DG].

[67] U. Schreiber, “From loop space mechanics to nonAbelian strings”, hep-th/0509163.

[68] T. Nikolaus, U. Schreiber and D. Stevenson “Principal infinity-bundles - General theory”, arXiv:1207.0248 [math.AT].

[69] T. Nikolaus, U. Schreiber and D. Stevenson “Principal infinity-bundles - Presentations”, arXiv:1207.0249 [math.AT].

[70] D. Roytenberg, “AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories”, Lett. Math. Phys. **79** (2007) 143 [hep-th/0608150].